Exact First Order Differential Equations

This Lecture covers material in Section 2.6. A first order differential equations is **exact** if it can be written in the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0,$$

where

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

Before showing how to solve these we need to review some multi-variable calculus, especially the **two-variable chain rule**. This will also help to motivate why equations of this form are important in physics.

Let $\psi(x,y)$ be a function of two variables. Then we can think of

$$z=\psi(x,y)$$

as a surface in three-dimensional space where z is the height above the xy-plane. Now suppose the x and y are functions of t (time) so that $(x(t),y(t))$ gives a curve in the xy-plane. Then $z(t)=\psi(x(t),y(t))$ gives a curve in three-dimensional space. Suppose we desire to know the rate of change of z with respect to t. According to the two-variable chain rule the answer is

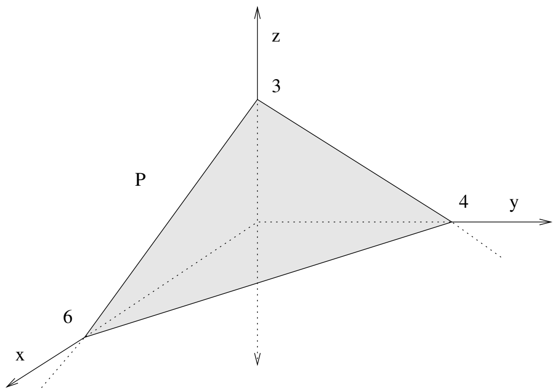
$$\frac{dz}{dt}=\frac{\partial\psi}{\partial x}\frac{dx}{dt}+\frac{\partial\psi}{\partial y}\frac{dy}{dt}. \qquad (*)$$

This formula is derived in Calculus III. Here I will give an intuitive motivation for why it works.

Suppose $\psi(x,y)$ is just a plane. Then we have

$$z=\psi(x,y)=Ax+By+C$$

for some constants A, B and C. Here A is the slope of the plane with respect to the x direction, B is the slope of the plane with respect to the y direction and C is the intercept with the z-axis.



$$z = -\tfrac{1}{2}x - \tfrac{3}{4}y + 3$$

We want to compute the change in z as t changes from t_0 to $t_0+\Delta t$. Let $\Delta x = x(t_0 + \Delta t) - x(t_0)$ and $\Delta y = y(t_0 + \Delta t) - y(t_0)$ be the changes in x and y, respectively. For convenience let $x_0 = x(t_0)$ and $y_0 = y(t_0)$. Then the change in z is

$$\Delta z = \psi(x_0 + \Delta x, y_0 + \Delta y) - \psi(x_0, y_0) = A\Delta x + B\Delta y.$$

We divide both sides by Δt to obtain

$$\frac{\Delta z}{\Delta t} = A\frac{\Delta x}{\Delta t} + B\frac{\Delta y}{\Delta t}.$$

Now we can find the derivative of z with respect to t by taking limits as $\Delta t \to 0$. This gives

$$\frac{dz}{dt} = A\frac{dx}{dt} + B\frac{dy}{dt}.$$

But notice that $A = \partial_x\psi$ and $B = \partial_y\psi$. Thus we have

$$\frac{dz}{dt} = \frac{\partial\psi}{\partial x}\frac{dx}{dt} + \frac{\partial\psi}{\partial y}\frac{dy}{dt}$$

which is $(*)$.

This shows that the two-variable chain rule works for planes. In general if $\psi(x, y)$ is reasonably smooth it can be approximated near each point by a tangent plane. It can be shown that this gives the two-variable chain rule for any function of two variables that is smooth enough that its graph has a tangent plane at each point in an open set containing the point of interest.

Now, let $z = \psi(x, y)$ be a surface. But suppose z is some quantity that is conserved, like energy. That is we now have

$$\psi(x, y) = C.$$

The slice of the surface through $z = C$ is called **level curve**.

Example: Let $z = \psi(x, y) = x^2 + y^2$. Then the level curve for $z = 1$ is a circle of radius 1 that floats one unit above the xy-plane.

Example: Let $z = \psi(x, y) = 3x + y - 3$. The level curve for $z = 2$ is the line $3x + y - 3 = 2$, or $y = -3x + 5$, that is it floating two units above the xy-plane.

For now suppose y is a function of x (at least implicitly). As we change x we cause y to change so that $z = \psi(x, y)$ stays on the same level curve. Since z is not changing we have $dz/dx = 0$. The two-variable chain rule, using x for t, gives

$$0 = \frac{dz}{dx} = \frac{\psi(x, y(x))}{dx} = \frac{\partial \psi}{\partial x}\frac{dx}{dx} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}.$$

Therefore,

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} y' = 0.$$

If ψ_x and ψ_y are known functions what we have is a differential equation in y.

We will be doing the inverse of this process. That is, given at differential equation in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

we will solve it for $y(x)$ (or at least a relation between x and y) by finding a surface $\psi(x, y)$ such that $M = \psi_x$ and $N = \psi_y$, and then using an initial condition to find the desired level curve. In many applications $\langle M, N\rangle$ is given as a force field and then ψ is a potential energy function. If energy is conserved, the dynamics are restricted to a level curve of $z = \psi(x, y)$.

Enough talk, let's do some examples.

Example 1. Solve $(2x + y) + (x + 2y)y' = 0$, with $y(3) = 1$.

Solution. We want to find a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = 2x + y \qquad \& \qquad \frac{\partial \psi}{\partial y} = x + 2y.$$

So, we integrate.

$$\psi = \int \psi_x \, dx = \int 2x + y \, dx = x^2 + xy + C_1(y),$$

where $C_1(y)$ an arbitrary function of y. The idea is we are finding the class of all functions whose partial derivative with respect to x gives $2x + y$.

But we also need for $\psi_y = x + 2y$. So, we integrate.

$$\psi = \int \psi_y \, dy = \int x + 2y \, dy = xy + y^2 + C_2(x),$$

where $C_2(x)$ can be any function of x.

We now have two classes of functions, each satisfying one of the two conditions. If we could find a function that is in both class that would do the trick. The answer is obvious. Let

$$\psi(x, y) = x^2 + xy + y^2.$$

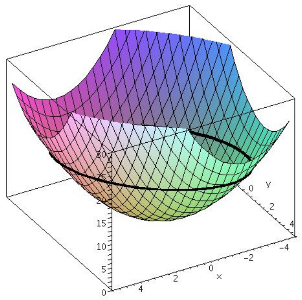
This function is in both classes and thus satisfies both the needed conditions. Now we consider the initial condition, $y(3) = 1$, that is, $x = 3 \implies y = 1$. Then

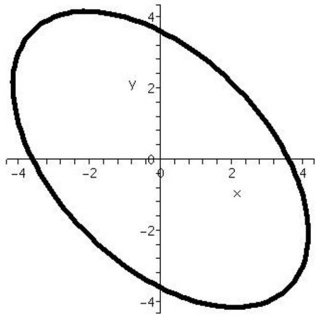
$$\psi(3, 1) = 9 + 3 + 1 = 13.$$

Thus, the level curve we want is

$$x^2 + xy + y^2 = 13.$$

We will leave as a relation. Below are plots of the surface $z = x^2 + xy + y^2$ with the level 13 curve and a projection of this curve into the xy-plane. $\square$





Extra Credit. Prove that this curve $x^2 + xy + y^2 = 13$ is an ellipse and find its focal points. You can do this by reviewing how to rotate graphs with rotation matrices and the properties of ellipses. Then rotate the graph 45^o so that its major axis lies along the x-axis.

Example 2 (Not!). Solve $(2x + 2y) + (x + 2y)y' = 0$, with $y(3) = 1$. We integrate.

$$\psi = \int 2x + 2y\, dx = x^2 + 2xy + C_1(y).$$

$$\psi = \int x + 2y\, dy = xy + y^2 + C_2(x).$$

Now look closely. Since $2xy \neq xy$ there is no function that meets both conditions. The method fails! What this means in physical terms is that the force field $\langle 2x + 2y, x + 2y \rangle$ does not arise from a potential function; in such a system energy is **not conserved**.

Note: This example can be converted to a separable equation because it is homogeneous. We work through this in the Appendix at the end of these notes. It is optional reading.

What we need is a quick test to see if ψ exists for a given equation so that we don't waste a lot of time barking up the wrong tree.

Theorem! Given two functions $M(x,y)$ and $N(x,y)$, there exists a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \qquad \& \qquad \frac{\partial \psi}{\partial y} = N(x,y),$$

if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

in an open rectangle containing the point of interest.

Check this for the two examples above. This is Theorem 2.6.1 in your textbook. Your textbook gives a proof, but another prove is covered in Calculus III that uses Green's Theorem. If you are a Math major read both and compare them. However, one direction is easy: if ψ exists, then $\psi_{xy} = \psi_{yx} \implies M_y = N_x$. This theorem is the motivation for the definition we gave at the beginning of an exact first order differential equation.

Example 3. Find the general solution to

$$y\cos x + ye^{xy} + (\sin x + xe^{xy})y' = 0.$$

Solution. Let $M = y\cos x + ye^{xy}$ and $N = \sin x + xe^{xy}$. Then

$$M_y = \cos x + e^{xy} + xye^{xy} = N_x.$$

Thus, it is exact. We integrate.

$$\psi = \int M\,dx = y\sin x + e^{xy} + C_1(y)$$

and

$$\psi = \int N\,dy = y\sin x + e^{xy} + C_2(x).$$

We let $\psi(x,y) = y\sin x + e^{xy}$. The general solution is then

$$y\sin x + e^{xy} = C.$$

□

Example 4. Solve $4x^3 + 4y^3y' = 0$, with $y(1) = 1$.

Solution. It is exact since $(4x^3)_y = 0$ and $(4y^3)_x = 0$. Then

$$\psi = \int 4x^3\,dx = x^4 + C_1(y)$$

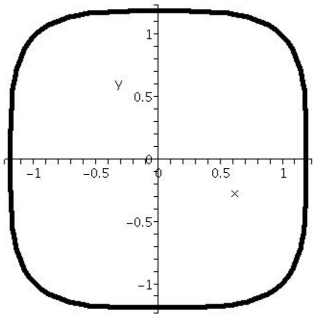
and

$$\psi = \int 4y^3\,dy = y^4 + C_2(x).$$

We let $\psi = x^4 + y^4$. Since (1,1) is our initial condition we see that our solution is

$$x^4 + y^4 = 2.$$

Below is a graph of this curve projected into the xy-plane. ☐



Example 5. Solve $4x^4 + 4xy^3y' = 0$, with $y(1) = 1$.

Solution. We check for exactness. $(4x^4)_y = 0$ while $(4xy^3)_x = 4y^3$. Thus it is not exact. But wait! Notice that this example is exactly the same as Example 4, but that we have multiplied through by x. So, if we now multiple through by $\frac{1}{x}$ we get

$$4x^3 + 4y^3y' = 0.$$

Thus, the solution is same as in Example 4! ☐

Integrating factors.

This last example motivates the following idea. Suppose we have a differential equation of the form

$$M + Ny' = 0$$

which is not exact. Can we find a function $\mu(x, y)$ such that

$$\mu M + \mu N y' = 0$$

is exact?

The answer is, not always, but sometimes you can. When this works we call μ an **integrating factor**. Finding such as μ can be tricky. Here we show three special cases where an integrating factor μ can be found. Each relies on an assumption about μ that can be tested for.

Case 1. Suppose a suitable μ exists and that it is a function of x only.

Case 2. Suppose a suitable μ exists and that it is a function of y only.

Case 3. Suppose a suitable μ exists and that it can be written as a function dependent only on the product xy.

In all cases we need to find μ such that

$$(\mu M)_y = (\mu N)_x$$

so that we have exactness. By the product rule this is equivalent to requiring

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x. \qquad (*)$$

Case 1. If Case 1 holds then $\mu_y = 0$ and we can think of μ_x as μ'. Then $(*)$ becomes

$$\mu M_y = \mu' N + \mu N_x$$

or

$$\frac{\mu'}{\mu} = \frac{M_y - N_x}{N}.$$

If our assumption is correct then, since μ'/μ depends only on x, we know that $(M_y - N_x)/N$ depends only on x. Then the integrals below are well defined.

$$\int \frac{1}{\mu}\, d\mu = \int \frac{M_y - N_x}{N}\, dx$$

Thus,

$$\mu = e^{\int \frac{M_y - N_x}{N}\, dx}.$$

In fact, this gives us a test to determine when this method will work. If $\frac{M_y - N_x}{N}$ depends only on x it follows that μ depends only on x.

Example 6. Find the general solution to $y^2 + x^3 + xyy' = 0$.

Solution. Since $(y^2 + x^3)_y = 2y$ and $(xy)_x = y$ are not equal, this equation is not exact. But

$$\frac{2y - y}{xy} = \frac{1}{x}$$

depends only on x. Thus we let

$$\mu = e^{\int \frac{1}{x}\, dx} = x.$$

So, we multiply through by x to get

$$xy^2 + x^4 + x^2yy' = 0.$$

Let $M = xy^2 + x^4$ and $N = x^2y$. Then $M_y = 2xy = N_x$, so we have exactness. Now we find ψ as before.

$$\psi = \int M\, dx = \frac{1}{2}x^2y^2 + \frac{1}{5}x^5 + C_1(y)$$

$$\psi = \int N\, dy = \frac{1}{2}x^2y^2 + C_2(x)$$

Thus, we let $\psi = \frac{1}{2}x^2y^2 + \frac{1}{5}x^5$, so the general solution is

$$\frac{1}{2}x^2y^2 + \frac{1}{5}x^5 = C,$$

or if you prefer

$$5x^2y^2 + 2x^5 = C.$$

Solving for y gives

$$y = \pm\sqrt{\frac{C - 2x^5}{5x^2}}.$$

□

Case 2. This is so similar to Case 1 that we leave it to you to develop the method and find the formula for $\mu(y)$.

Case 3. Recall equation $(*)$: $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. Let $v = xy$ and remember we are assuming μ can be rewritten as a function of v. Thus,

$$\mu_y = \frac{\partial \mu(v)}{\partial y} = \frac{d\mu}{dv}\frac{\partial v}{\partial y} = \frac{d\mu}{dv} \cdot x = x\mu',$$

and

$$\mu_x = \frac{\partial \mu(v)}{\partial x} = \frac{d\mu}{dv}\frac{\partial v}{\partial y} = \frac{d\mu}{dv} \cdot y = y\mu',$$

where μ' means the derivative with respect to v. Now $(*)$ becomes

$$x\mu' M + \mu M_y = y\mu' N + \mu N_x,$$

which gives

$$\frac{\mu'}{\mu} = \frac{N_x - M_y}{xM - yN}.$$

If the right hand side depends only on $v = xy$ then the assumption we are making is valid, and thus

$$\mu = e^{\int \frac{N_x - M_y}{xM - yN}\,dv}.$$

Perhaps an example would help.

Example 7. Solve

$$5x^3 + \frac{1}{x}\cos xy + \frac{x^4 + \cos xy}{y}\frac{dy}{dx} = 0,$$

with $y(1) = \pi$.

Solution. Let $M = 5x^3 + \frac{1}{x}\cos xy$ and $N = \frac{x^4+\cos xy}{y}$. Then

$$M_y = -\sin xy \qquad \& \qquad N_x = \frac{4x^3 - y\sin xy}{y}$$

Thus, the given equation is not exact. We now search for an integration factor.

$$\text{Case 1. } \frac{M_y - N_x}{N} = \frac{-\frac{4x^3}{y}}{\frac{x^4+\cos xy}{y}} = \frac{-4x^3}{x^4 + \cos xy} \quad \text{No good!}$$

$$\text{Case 2. } \frac{N_x - M_y}{M} = \frac{\frac{4x^3}{y}}{\frac{5x^4+\cos xy}{x}} = \frac{4x^4}{5x^4 y + y\cos xy} \quad \text{Rats!!}$$

$$\text{Case 3. } \frac{N_x - M_y}{xM - yN} = \frac{\frac{4x^3}{y}}{5x^4 + \cos xy - (x^4 + \cos xy)} = \frac{1}{xy}! \text{ Eureka!!!}$$

Let $v = xy$. Now,

$$\mu(v) = e^{\int \frac{1}{v}\,dv} = e^{\ln|v| + C} = C|v| = C|xy|;$$

we will use $\mu = xy$

On ward! We multiply the original equation by xy to get

$$5x^4 y + y\cos xy + (x^5 + x\cos xy)y' = 0.$$

Let $M = 5x^4 y + y\cos xy$ and $N = x^5 + x\cos xy$. We double check that it is in fact exact.

$$M_y = 5x^4 + \cos xy - xy\sin xy = N_x.$$

Now the hunt is on for ψ!

$$\psi = \int M\,dx = x^5 y + \sin xy + C_1(y)$$

and

$$\psi = \int N\,dy = x^5 y + \sin xy + C_2(x).$$

Thus, $\psi = x^5 y + \sin xy$ and the general solution is $x^5 y + \sin xy = C$. Since $y(1) = \pi$ you can check that $C = \pi$. Thus, the solution is

$$x^5 y + \sin xy = \pi.$$

□

APPENDIX A: MORE ON EXAMPLE 2.
[OPTIONAL READING]

Recall that Example 2 was not exact, but that it was noted that it is homogeneous. Here we will solve it and study the solution.

$$y' = \frac{-2x - 2y}{x + 2y} = \frac{-2 - 2y/x}{1 + 2y/x} = \frac{-2 - 2v}{1 + 2v},$$

where v = y/x. It follows that $y' = v + xv'$. Now we have

$$x\frac{dv}{dx} = \frac{-2 - 2v}{1 + 2v} - v = \frac{-2 - 2v}{1 + 2v} - v\frac{1 + 2v}{1 + 2v} = \frac{-2 - 3v - 2v^2}{1 + 2v}.$$

Thus,

$$-\int \frac{1 + 2v}{2 + 3v + 2v^2}\,dv = \int \frac{1}{x}\,dx.$$

We rewrite the left integrand as

$$\frac{1}{2}\frac{4v + 3 - 1}{2v^2 + 3v + 2} = \frac{1}{2}\left(\frac{4v + 3}{2v^2 + 3v + 2} - \frac{1}{2v^2 + 3v + 2}\right).$$

Now,

$$\int \frac{4v + 3}{2v^2 + 3v + 2}\,dv = \ln|2v^2 + 3v + 2| + C,$$

and

$$\int \frac{1}{2v^2+3v+2}\,dv = \int \frac{1}{\left(\sqrt{2}v+\frac{3}{2\sqrt{2}}\right)^2+7/8}\,dv.$$

Let $u=\sqrt{2}v+\frac{3}{2\sqrt{2}}$. Then $du=\sqrt{2}dv$. The last integral becomes

$$\frac{1}{\sqrt{2}}\int \frac{du}{u^2+7/8} = \frac{4\sqrt{2}}{7}\int \frac{du}{8u^2/7+1}.$$

Next let $w=\frac{2\sqrt{2}u}{\sqrt{7}}$. Then $dw=\frac{2\sqrt{2}du}{\sqrt{7}}$. Thus, the last integral becomes

$$\frac{2}{\sqrt{7}}\int \frac{dw}{w^2+1} = \frac{2}{\sqrt{7}}\arctan(w)+C.$$

Putting all this together gives

$$\ln|x|+C = -\frac{1}{2}\ln|2v^2+3v+2|+\frac{1}{\sqrt{7}}\arctan\left(\frac{4v+3}{\sqrt{7}}\right).$$

Now we find C, Recall we were given $y(1)=1$. Since $v=y/x$ we get $v=1$. Thus,

$$0+C=\ln\left(\frac{1}{\sqrt{7}}\right)+\frac{1}{\sqrt{7}}\arctan\left(\sqrt{7}\right).$$

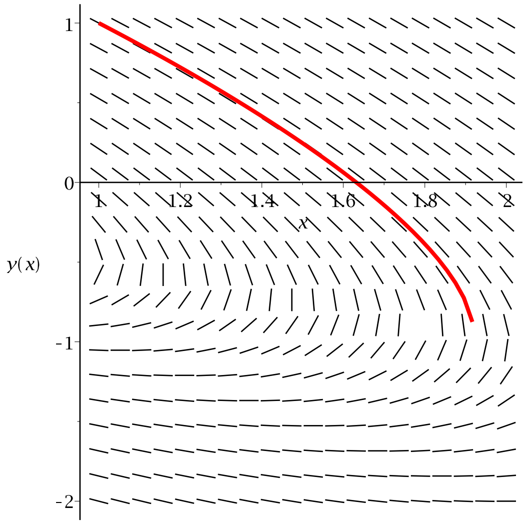
Thus, our solution is given by the relation

$$\ln x = -\frac{1}{2}\ln(2v^2+3v+2)+\frac{1}{\sqrt{7}}\arctan\left(\frac{4v+3}{\sqrt{7}}\right) \\ -\ln\left(\frac{1}{\sqrt{7}}\right)-\frac{1}{\sqrt{7}}\arctan\left(\sqrt{7}\right),$$

where $v=y/x$.

I tried to plot this, and some simplifications, using `implicitplot` in Maple 17 and using WolframAlpha online. Neither could do anything with it. So, I punted and used Maple to numerically plot the solution curve. Here is the command and the plot it produced.

```
>DEplot(diff(y(x),x)=(-2*x-2*y(x))/(x+2*y(x)),y(x),
x=1.0..2.0,[[y(1)=1]],y=-2.0..1.0,arrows=line,color=black,
linecolor=red);
```



It also produced this warning: “Warning, plot may be incomplete, the following errors(s) were issued: cannot evaluate the solution further right of 1.9201778, probably a singularity”.

If you look at the original differential equation you may notice that $y'(x)$ is undefined when $x = -2y$. Study the graph. At x equal about 2, y is just about -1. Our solution curve ends and is not valid passed this point.