Homogeneous Second Order Differential Equations with Constant Coefficients: Continued

Complex roots.

The final case is what to do when the roots of the characteristic polynomial are complex. Recall that this means they will be of the form $p \pm qi$ for real numbers p and q where $i^2 = -1$. (This assumes that the coefficients a, b and c are real.) We will need to "review" some facts about complex functions that where censored from your calculus textbook.

But first, let's look at a simple example, $y'' + y = 0$. We can rewrite this as $y'' = -y$. So, we are seeking functions whose second derivatives are their own negatives. Two might come to mind, $\sin x$ and $\cos x$. In fact $y = C_1 \sin x + C_2 \cos x$ gives all possible solutions, as we will show later.

The roots of the characteristic polynomial, $r^2 + 1 = 0$, are $\pm i$. Notice that

$$(e^{ix})'' = (ie^{ix})' = i^2 e^{ix} = -e^{ix}.$$

But, what does it mean to raise e to a complex power? And, what does it mean to take a derivative of such a function? And, how are these functions connected to $\sin x$ and $\cos x$?

Way back in Calculus II you studied Taylor series and you learned that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

Suppose $z = a + ib$ is a complex number. Then we define

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots .$$

In courses on Complex Analysis (MATH 455 here) it is shown that this sequence converges for all complex numbers z. The derivative can be defined via term-by-term differentiation. The following facts can also be proven:

$$e^{a+ib} = e^a e^{ib}$$

$$\frac{de^{\alpha ix}}{dx} = \alpha i e^{\alpha ix},$$

for any real (or complex) number α. Now watch.

$$\begin{aligned} e^{ix} &= 1+ix+\frac{(ix)^2}{2}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\frac{(ix)^6}{6!}+\frac{(ix)^7}{7!}+\frac{(ix)^8}{8!}+\cdots \\ &= 1+ix-\frac{x^2}{2}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\frac{x^6}{6!}-i\frac{x^7}{7!}+\frac{x^8}{8!}+\cdots \\ &= \left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}+\cdots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right) \\ &= \cos x + i\sin x. \end{aligned}$$

Now let's get back to differential equations. Suppose we have $ay''+by'+cy=0$ and the roots of ar^2+br+c are $\alpha \pm i\beta$. Then the general solution is

$$\begin{aligned} y &= C_1e^{(\alpha+i\beta)x}+C_2e^{(\alpha-i\beta)x} = (C_1e^{i\beta x}+C_2e^{-i\beta x})e^{\alpha x} \\ &= (C_1(\cos(\beta x)+i\sin(\beta x))+C_2(\cos(\beta x)-i\sin(\beta x)))e^{\alpha x} \\ &= ((C_1+C_2)\cos(\beta x)+i(C_1-C_2)\sin(\beta x))\,e^{\alpha x}. \end{aligned}$$

We can rewrite this as

$$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x.$$

Theorem 3. The general solution to $ay''+by'+cy=0$ when the roots of the characteristic polynomial are $\alpha \pm i\beta$ is

$$y = Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x.$$

If $y(x_0)=p$ and $y'(x_0)=q$ then there is a unique solution for A and B.

Proof. We have already derived the solution, but you can check it by directly substituting it in to the differential equation. Next, $y(x_0)=p$ implies

$$A\cos\beta x_0 + B\sin\beta x_0 = pe^{-\alpha x_0}.$$

And $y'(x_0) = q$ implies

$$A\alpha e^{\alpha x_0}\cos\beta x_0 - A\beta e^{\alpha x_0}\sin\beta x_0 + B\alpha e^{\alpha x_0}\sin\beta x_0 + B\beta e^{\alpha x_0}\cos\beta x_0 = q,$$

or

$$A(\alpha\cos\beta x_0 - \beta\sin\beta x_0) + B(\beta\cos\beta x_0 + \alpha\sin\beta x_0) = qe^{-\alpha x_0}.$$

So, again we have two equations and two unknowns and these can readily by solved for A and B. □

Example. Find the general solution to $y'' - y' + 2y = 0$. Then find the solution for the initial values $y(0) = p$, $y'(0) = q$.

Solution. The characteristic polynomial $r^2 - r + 2$ has complex roots $r = \frac{1}{2} \pm i\frac{\sqrt{7}}{2}$. Thus, the general solution is

$$y(x) = Ae^{\frac{1}{2}x}\cos\frac{\sqrt{7}}{2}x + Be^{\frac{1}{2}x}\sin\frac{\sqrt{7}}{2}x.$$

Now, $y(0) = p$ implies $A = p$ and $y'(0) = q$ gives $p/2 + B\sqrt{7}/2 = q$. Thus, $B = \frac{2q-p}{\sqrt{7}}$ and we have

$$y(x) = pe^{\frac{1}{2}x}\cos\frac{\sqrt{7}}{2}x + \frac{2q-p}{\sqrt{7}}e^{\frac{1}{2}x}\sin\frac{\sqrt{7}}{2}x.$$

□

Summary

Given $ay''(x) + by'(x) + cy(x) = 0$ we have three cases. These depend on the roots of the characteristic polynomial $ar^2 + br + c = 0$.

Case 1. The roots of the characteristic polynomial, r_1 and r_2, are real and distinct, that is $r_1 \neq r_2$. Then the general solution is

$$y(x) = C_1e^{r_1x} + C_2e^{r_2x}.$$

Case 2. The characteristic polynomial has a single real root, r. Then the general solution is

$$y(x) = C_1e^{rx} + C_2xe^{rx}.$$

Case 3. The roots of the characteristic polynomial are complex conjugates, $\alpha \pm \beta$. Then the general solution is

$$y(x) = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x.$$

In each case we can find unique values of C_1 and C_2 for any given pair of initial conditions of the form

$$y(x_0) = y_0 \ \ \& \ \ y'(x_0) = v_0.$$