Lecture Notes for Ch 10
Fourier Series and Partial Differential Equations

Part I.
Outline

Page 2. Review even and odd functions. Very important.


Page 4. Definition of Fourier series and a Theorem.

Pages 5-8. Verification of parts of the Theorem.

Pages 9-10. Fourier series of a Square Wave.

Pages 11-12. Fourier series of a Triangle Wave.
Review of Even and Odd Functions

Even: \( f(-x) = f(x) \)  
Odd: \( f(-x) = -f(x) \).

Even Examples: \( 5x^6 - 3x^4 + 2, \frac{x^4+1}{x^2+1}, |x|, \cos(x), \cos(x^5), \sin^4(x). \)

Odd Examples: \( x^{1/3}, x^7 - 6x^3, |x|, \sin(x), \sin^5(x^7). \)

Neither: \( x^2 + x, x + \cos(x), \frac{1}{x+1}. \)

Three Even Functions

Three Odd Functions

Problem: Let \( f(x) \) be an odd function and suppose it is defined at \( x = 0 \). What is \( f(0) \)? Prove this!

Problem: Find a function with domain the whole real line that is both even and odd. (There is only one correct answer.)

Fact: If \( f(x) \) is even, then \( \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx. \)

Fact: If \( f(x) \) is odd, then \( \int_{-a}^{a} f(x) \, dx = 0. \) Draw pictures to see intuitively why these two facts hold.

Fact: If \( f(x) \) is even and differentiable, then \( f'(x) \) is odd, and vice versa. This can be proved with the Chain Rule. Suppose \( f(x) \) is even and differentiable. Then \( (f(-x))' = f'(-x)(-1). \) But \( f(-x) = f(x), \) so \( (f(-x))' = (f(x))' = f'(x). \) Thus, \( f'(-x)(-1) = f'(x), \) or \( f'(-x) = -f'(x). \) You can also draw pictures of tangent lines to curves to see this intuitively.

Exercises: Suppose that \( f(x) \) and \( g(x) \) are even and the \( h(x) \) and \( k(x) \) are odd. Then show that:

(a) \( f(x)g(x) \) is even.  (f) \( f(g(x)) \) is even.  
(b) \( f(x)h(x) \) is odd.  (g) \( f(h(x)) \) is even.  
(c) \( h(x)k(x) \) is even.  (h) \( h(f(x)) \) is even.  
(d) \( f(x) + g(x) \) is even.  (i) \( h(k(x)) \) is odd.  
(e) \( h(x) + k(x) \) is odd.  (j) \( f(x) + h(x) \) need not be even or odd.

Example: Let \( p(x) = f(x)h(x). \) Then \( p(-x) = f(-x)h(-x) = f(x)(-h(x)) = -f(x)h(x) = -p(x). \)
Hence we have an odd function.

You may be tested on this!
Some Damn Useful Integral Formulas

1. \[ \int_{-L}^{L} \cos \left(\frac{m\pi x}{L}\right) \cos \left(\frac{n\pi x}{L}\right) \, dx = \begin{cases} 0 & \text{for } m \neq n, \\ L & \text{for } m = n. \end{cases} \]

2. \[ \int_{-L}^{L} \cos \left(\frac{m\pi x}{L}\right) \sin \left(\frac{n\pi x}{L}\right) \, dx = 0, \] for all integers \( m \) and \( n \)

3. \[ \int_{-L}^{L} \sin \left(\frac{m\pi x}{L}\right) \sin \left(\frac{n\pi x}{L}\right) \, dx = \begin{cases} 0 & \text{for } m \neq n, \\ L & \text{for } m = n. \end{cases} \]

Below and to the left are the overlaid plots of \( \cos 4\pi x \) and \( \cos 2\pi x \). Below and to the right is the graph of their product. Study this. Make some similar plots on your own until the formulas above make sense.

We will prove only a special case of \#1. Let \( L = \pi \), \( m = 3 \) and \( n = 2 \). The proof uses the trig identities

\[ \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \]

and its corollary

\[ \cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi \]

Thus,

\[ \cos \theta \cos \phi = \frac{1}{2} (\cos(\theta + \phi) + \cos(\theta - \phi)). \]

Applying this to our case gives

\[ \cos 3x \cos 2x = \frac{1}{2} (\cos 5x + \cos x). \]

Thus,

\[ \int_{-\pi}^{\pi} \cos 3x \cos 2x \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos 5x + \cos x \, dx = 0 + 0 = 0. \]

Let’s consider one more special case: \( L = \pi \), \( m = n = 2 \). Then

\[ \int_{-\pi}^{\pi} \cos^2 2x \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos 4x + \cos 0 \, dx = \frac{0 + 2\pi}{2} = \pi. \]

From these ideas you should be able to derive the three integrals formulas. If you have had linear algebra, you might notice that the integral of a product of two functions is a kind of inner product and so the cosine and sine functions used above are mutually orthogonal.
Definition of Fourier Series

Let \( f(x) \) be a piecewise continuous periodic function with period \( 2L \). The Fourier Series of \( f(x) \) is,

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right),
\]

where

\[
a_n = \frac{1}{L} \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) f(x) \, dx, \quad n = 0, 1, 2, 3, \ldots,
\]

and

\[
b_n = \frac{1}{L} \int_{-L}^{L} \sin \left( \frac{n\pi x}{L} \right) f(x) \, dx, \quad n = 1, 2, 3, \ldots.
\]

**Theorem.** The Fourier series of \( f(x) \) converges to \( f(x) \) if \( f \) is continuous at \( x \). If \( f \) is discontinuous at \( x \) then this must be a jump discontinuity. Let

\[
f(x^+) = \lim_{c \to x^+} f(c) \quad \text{and} \quad f(x^-) = \lim_{c \to x^-} f(c).
\]

Then the Fourier series of \( f \) converges to the average of these two limits,

\[
\frac{f(x^+) + f(x^-)}{2}.
\]

If \( f \) is an even function then \( b_n = 0 \), for \( n = 1, 2, 3, \ldots \).

If \( f \) is an odd function then \( a_n = 0 \), for \( n = 0, 1, 2, 3, \ldots \).

In all cases \( a_0/2 \) is the average value of \( f(x) \) over one period.

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We will verify parts of this theorem. The full proof is covered in Math 407.

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1 Technical note: It is also required that \( f(x) \) is bounded and that in each period of \( f(x) \) there are only finitely many extrema.
We verify the formula for $a_0$ and show it is the average value of $f(x)$.

Let $c_n = \cos\left(\frac{n\pi x}{L}\right)$ and $s_n = \sin\left(\frac{n\pi x}{L}\right)$.

Suppose

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} b_n s_n.$$ 

We will show that

$$a_0 = \frac{1}{L} \int_{-L}^{L} \cos(0) f(x) \, dx.$$ 

$$\frac{1}{L} \int_{-L}^{L} \cos(0) f(x) \, dx = \frac{1}{L} \int_{-L}^{L} 1 \cdot f(x) \, dx$$ 

$$= \frac{1}{L} \int_{-L}^{L} \left( \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} b_n s_n \right) \, dx$$

$$= \frac{1}{L} \int_{-L}^{L} \frac{a_0}{2} \, dx + \frac{1}{L} \sum_{n=1}^{\infty} a_n \int_{-L}^{L} c_n \, dx + \frac{1}{L} \sum_{n=1}^{\infty} b_n \int_{-L}^{L} s_n \, dx$$

$$= a_0 + \frac{1}{L} \sum_{n=1}^{\infty} a_n \cdot 0 + \frac{1}{L} \sum_{n=1}^{\infty} b_n \cdot 0 = a_0$$ 

since,

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \, dx = \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \bigg|_{-L}^{L} = 0 - 0 = 0,$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \, dx = -\frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right) \bigg|_{-L}^{L} = -\frac{L}{n\pi} (\cos(n\pi) - \cos(-n\pi)) = 0,$$

and

$$\frac{1}{L} \int_{-L}^{L} \frac{a_0}{2} \, dx = \frac{a_0}{2L} \int_{-L}^{L} 1 \, dx = \frac{a_0}{2L} \cdot 2L = a_0.$$ 

Finally, recall that the average value of $f(x)$ over a cycle is $\frac{1}{2L} \int_{-L}^{L} f(x) \, dx$. Thus, $a_0/2$ is the average value of $f(x)$ over a cycle.
We check the formula for $a_7$, but the method is the same for all $n \geq 1$.

\[
a_7 = \frac{1}{L} \int_{-L}^{L} \cos \left( \frac{7\pi x}{L} \right) f(x) \, dx
\]

\[= \frac{a_0}{2L} \int_{-L}^{L} c_7 \, dx + \frac{1}{L} \sum_{n=1}^{\infty} a_n \int_{-L}^{L} c_n c_7 \, dx + \frac{1}{L} \sum_{n=1}^{\infty} b_n \int_{-L}^{L} s_n c_7 \, dx\]

\[= \frac{a_0}{2L} \cdot 0 + \frac{1}{L}(0 + 0 + 0 + 0 + 0 + a_7 L + 0 + 0 + \cdots) + \frac{1}{L}(0 + 0 + 0 + \cdots)
\]

\[= a_7.
\]

Next we check $b_4$.

\[
b_4 = \frac{1}{L} \int_{-L}^{L} \sin \left( \frac{4\pi x}{L} \right) f(x) \, dx
\]

\[= \frac{a_0}{2L} \int_{-L}^{L} s_4 \, dx + \frac{1}{L} \sum_{n=1}^{\infty} a_n \int_{-L}^{L} c_n s_4 \, dx + \frac{1}{L} \sum_{n=1}^{\infty} b_n \int_{-L}^{L} s_n s_4 \, dx\]

\[= \frac{a_0}{2L} \cdot 0 + \frac{1}{L}(0 + 0 + 0 + \cdots) + \frac{1}{L}(0 + 0 + b_4 L + 0 + 0 + \cdots)
\]

\[= b_4.
\]
Even and Odd Properties

If $f(x)$ is an odd function then

$$a_n = \frac{1}{L} \int_{-L}^{L} \cos \left( \frac{n\pi x}{L} \right) f(x) \, dx = 0,$$

since the product of an even function with an odd function is odd.

If $f(x)$ is an even function then

$$b_n = \frac{1}{L} \int_{-L}^{L} \sin \left( \frac{n\pi x}{L} \right) f(x) \, dx = 0,$$

since the product of an odd function with an even function is odd.
An Example: The Square Wave

Let

\[ f(x) = \begin{cases} 
-1 & \text{for } -\pi < x < 0 \\
1 & \text{for } 0 < x < \pi,
\end{cases} \]

with \( f(x + 2\pi) = f(x) \) for all \( x \).

Problem. Find the Fourier series of \( f(x) \).

Solution. Since \( f(x) \) is odd, \( a_n = 0 \) for \( n = 0, 1, 2, \ldots \).

Next,

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x) \, dx = \frac{2}{\pi} \int_{0}^{\pi} \sin(nx) \cdot 1 \, dx = \frac{-2}{n\pi} (\cos(n\pi) - \cos(0)) = \begin{cases} 
0 & \text{for } n \text{ even} \\
\frac{4}{n\pi} & \text{for } n \text{ odd}
\end{cases} \]

Thus,

\[ f(x) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin((2k-1)x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots. \]

Recall that using \( (2k-1) \) is a way to generate only odd numbers.

On the next page we plot a few partial sums.

Let \( f_N(x) = \sum_{k=1}^{N} \frac{4}{(2k-1)\pi} \sin((2k-1)x) \).
Plots of $f_N(x)$ for $N = 1, 2, 3, 4, 10, 100$

$N = 1$

$N = 2$

$N = 3$

$N = 4$

$N = 10$

$N = 100$
Another Example! A Triangle Wave

Problem: Find the Fourier series of the wave depicted below.

Solution: First note that since the period is 2, $L = 1$. We can see that for $0 < x < 1$, $f(x) = 1 - x$. Since this function is even, $b_n = 0$ for $n = 1, 2, 3, \ldots$. Also $\frac{a_0}{2} = \text{ave. value of } f(x) = \frac{1}{2}$. Thus, $a_0 = 1$.

Now,

\[
a_n = \frac{1}{1} \int_{-1}^{1} \cos(n\pi x) f(x) \, dx
\]

\[
= 2 \int_{0}^{1} \cos(n\pi x) (1 - x) \, dx
\]

\[
= \frac{2}{n\pi} \left[ \sin(n\pi x) - x \sin(n\pi x) - \frac{1}{n\pi} \cos(n\pi x) \right]_{0}^{1}
\]

\[
= \frac{2}{n\pi} \left[ \left( -\frac{1}{n\pi} \cos(n\pi) \right) - \left( -\frac{1}{n\pi} \right) \right]
\]

\[
= \begin{cases} 
    0 & \text{for even } n \\
    \frac{4}{n^2\pi^2} & \text{for odd } n.
\end{cases}
\]

Thus,

\[
f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x).
\]

On the next page we plot a few partial sums.

Let $f_N(x) = \frac{1}{2} + \sum_{k=1}^{N} \frac{4}{(2k-1)^2\pi^2} \cos((2k-1)\pi x)$. 
Plots of $f_N(x)$ for $N = 1, 2, 3, 30$