**Lecture Notes for Ch 10**
**Fourier Series and Partial Differential Equations**

**Part I.**
Outline

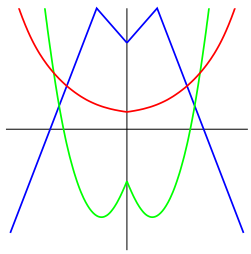
**Review of Even and Odd Functions**

| Even: $f(-x) = f(x)$ | Odd: $f(-x) = -f(x)$. |
|---|---|

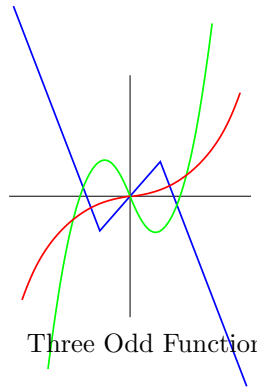
Even Examples: $5x^6 - 3x^4 + 2$, $\frac{x^4+1}{x^2-7}$, $|x|$, $\cos(x)$, $\cos(x^5)$, $\sin^4(x)$.

Odd Examples: $x^{1/3}$, $x^7 - 6x^3$, $x|x|$, $\sin(x)$, $\sin^5(x^7)$.

Neither: $x^2 + x$, $x + \cos(x)$, $\frac{1}{x+1}$.



Three Even Functions



Three Odd Functions

Problem: Let $f(x)$ be an odd function and suppose it is defined at $x = 0$. What is $f(0)$? Prove this!

Problem: Find a function with domain the whole real line that is both even and odd. (There is only one correct answer.)

Fact: If $f(x)$ is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Fact: If $f(x)$ is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$. Draw pictures to see intuitively why these two facts hold.

Fact: If $f(x)$ is even and differentiable, then $f'(x)$ is odd, and vice versa. This can be proved with the Chain Rule. Suppose $f(x)$ is even and differentiable. Then $(f(-x))' = f'(-x)(-1)$. But $f(-x) = f(x)$, so $(f(-x))' = (f(x))' = f'(x)$. Thus, $f'(-x)(-1) = f'(x)$, or $f'(-x) = -f'(x)$. You can also draw pictures of tangent lines to curves to see this intuitively.

Exercises: Suppose that $f(x)$ and $g(x)$ are even and the $h(x)$ and $k(x)$ are odd. Then show that:

| | |
|---|---|
| (a) $f(x)g(x)$ is even. | (f) $f(g(x))$ is even. |
| (b) $f(x)h(x)$ is odd. | (g) $f(h(x))$ is even. |
| (c) $h(x)k(x)$ is even. | (h) $h(f(x))$ is even. |
| (d) $f(x) + g(x)$ is even. | (i) $h(k(x))$ is odd. |
| (e) $h(x) + k(x)$ is odd. | (j) $f(x) + h(x)$ need not be even or odd. |

Example: Let $p(x) = f(x)h(x)$. Then $p(-x) = f(-x)h(-x) = f(x)(-h(x)) = -f(x)h(x) = -p(x)$. Hence we have an odd function.

**You may be tested on this!**

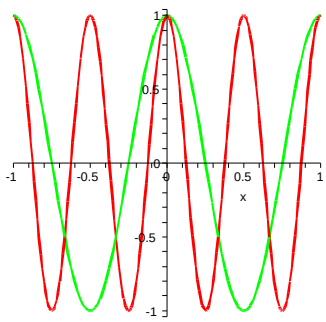
**Some Damn Useful Integral Formulas**

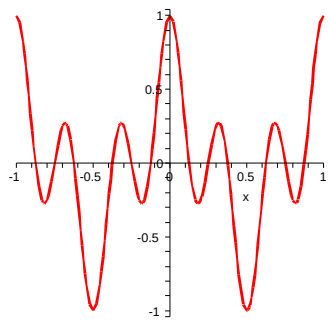
1. $\displaystyle\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases} 0 & \text{for} \quad m \neq n, \\ L & \text{for} \quad m = n. \end{cases}$

2. $\displaystyle\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx = 0,$ for all integers $m$ and $n$

3. $\displaystyle\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases} 0 & \text{for} \quad m \neq n, \\ L & \text{for} \quad m = n. \end{cases}$

Below and to the left are the overlaid plots of $\cos 4\pi x$ and $\cos 2\pi x$. Below and to the right is the graph of their product. Study this. Make some similar plots on your own until the formulas above make sense.




We will prove only a special case of #1. Let $L = \pi$, $m = 3$ and $n = 2$. The proof uses the trig identities

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

and its corollary

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

Thus,

$$\cos\theta\cos\phi = \frac{1}{2}\left(\cos(\theta + \phi) + \cos(\theta - \phi)\right).$$

Applying this to our case gives

$$\cos 3x\cos 2x = \frac{1}{2}\left(\cos 5x + \cos x\right).$$

Thus,

$$\int_{-\pi}^{\pi} \cos 3x\cos 2x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos 5x + \cos x\,dx = 0 + 0 = 0.$$

Let's consider one more special case: $L = \pi$, $m = n = 2$. Then

$$\int_{-\pi}^{\pi} \cos^2 2x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos 4x + \cos 0\,dx = \frac{0 + 2\pi}{2} = \pi.$$

From these ideas you should be able to derive the three integrals formulas. If you have had linear algebra, you might notice that the integral of a product of two functions is a kind of inner product and so the cosine and sine functions used above are mutually orthogonal.

**Definition of Fourier Series**

Let $f(x)$ be a piecewise continuous periodic function with period $2L$.[1] The Fourier Series of $f(x)$ is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx, \quad n = 0, 1, 2, 3, \ldots,$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx, \quad n = 1, 2, 3, \ldots.$$

**Theorem.** The Fourier series of $f(x)$ converges to $f(x)$ if $f$ is continuous at $x$. If $f$ is discontinuous at $x$ then this must be a jump discontinuity. Let

$$f(x^+) = \lim_{c \to x^+} f(c) \qquad \text{and} \qquad f(x^-) = \lim_{c \to x^-} f(c).$$

Then the Fourier series of $f$ converges to the average of these two limits,

$$\frac{f(x^+) + f(x^-)}{2}.$$

If $f$ is an even function then $b_n = 0$, for $n = 1, 2, 3, \ldots$.
If $f$ is an odd function then $a_n = 0$, for $n = 0, 1, 2, 3, \ldots$.
In all cases $a_0/2$ is the average value of $f(x)$ over one period.

We will verify parts of this theorem. The full proof is covered in Math 407.

[1] Technical note: It is also required that $f(x)$ is bounded and that in each period of $f(x)$ there are only finitely many extrema.

**We verify the formula for $\mathbf{a}_0$ and show it is the average value of $\mathbf{f}(x)$.**

Let $c_n = \cos\left(\frac{n\pi x}{L}\right)$ and $s_n = \sin\left(\frac{n\pi x}{L}\right)$.

Suppose

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} b_n s_n.$$

We will show that

$$a_0 = \frac{1}{L}\int_{-L}^{L} \cos(0) f(x)\, dx.$$

$$\begin{aligned}
\frac{1}{L}\int_{-L}^{L} \cos(0) f(x)\, dx &= \frac{1}{L}\int_{-L}^{L} 1 \cdot f(x)\, dx \\
&= \frac{1}{L}\int_{-L}^{L} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} b_n s_n\right) dx \\
&= \frac{1}{L}\int_{-L}^{L} \frac{a_0}{2}\, dx + \frac{1}{L}\sum_{n=1}^{\infty} a_n \int_{-L}^{L} c_n\, dx + \frac{1}{L}\sum_{n=1}^{\infty} b_n \int_{-L}^{L} s_n\, dx \\
&= a_0 + \frac{1}{L}\sum_{n=1}^{\infty} a_n \cdot 0 + \frac{1}{L}\sum_{n=1}^{\infty} b_n \cdot 0 = a_0
\end{aligned}$$

since,

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx = \frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} = 0 - 0 = 0,$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{-L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} = \frac{-L}{n\pi}\left(\cos(n\pi) - \cos(-n\pi)\right) = 0,$$

and

$$\frac{1}{L}\int_{-L}^{L} \frac{a_0}{2}\, dx = \frac{a_0}{2L}\int_{-L}^{L} 1\, dx = \frac{a_0}{2L} \cdot 2L = a_0.$$

Finally, recall that the average value of $f(x)$ over a cycle is $\frac{1}{2L}\int_{-L}^{L} f(x)\, dx$. Thus, $a_0/2$ is the average value of $f(x)$ over a cycle.

**We check the formula for $\mathbf{a_7}$,**
**but the method is the same for all $\mathbf{n \geq 1}$.**

$$
\begin{aligned}
a_7 &\stackrel{?}{=} \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{7\pi x}{L}\right) f(x)\,dx \\
&= \frac{1}{L}\int_{-L}^{L} c_7 \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} b_n s_n\right) dx \\
&= \frac{a_0}{2L}\int_{-L}^{L} c_7\,dx + \frac{1}{L}\sum_{n=1}^{\infty} a_n \int_{-L}^{L} c_n c_7\,dx + \frac{1}{L}\sum_{n=1}^{\infty} b_n \int_{-L}^{L} s_n c_7\,dx \\
&= \frac{a_0}{2L}\cdot 0 + \frac{1}{L}(0+0+0+0+0+0+a_7L+0+0+0+\cdots) + \frac{1}{L}(0+0+0+\cdots) \\
&= a_7.
\end{aligned}
$$

**Next we check $\mathbf{b_4}$.**

$$
\begin{aligned}
b_4 &\stackrel{?}{=} \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{4\pi x}{L}\right) f(x)\,dx \\
&= \frac{1}{L}\int_{-L}^{L} s_4 \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} b_n s_n\right) dx \\
&= \frac{a_0}{2L}\int_{-L}^{L} s_4\,dx + \frac{1}{L}\sum_{n=1}^{\infty} a_n \int_{-L}^{L} c_n s_4\,dx + \frac{1}{L}\sum_{n=1}^{\infty} b_n \int_{-L}^{L} s_n s_4\,dx \\
&= \frac{a_0}{2L}\cdot 0 + \frac{1}{L}(0+0+0+\cdots) + \frac{1}{L}(0+0+0+b_4L+0+0+0+\cdots) \\
&= b_4.
\end{aligned}
$$

**Even and Odd Properties**

If $f(x)$ is an odd function then

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\, dx = 0,$$

since the product of an even function with an odd function is odd.

If $f(x)$ is an even function then

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx = 0,$$

since the product of an odd function with an even function is odd.

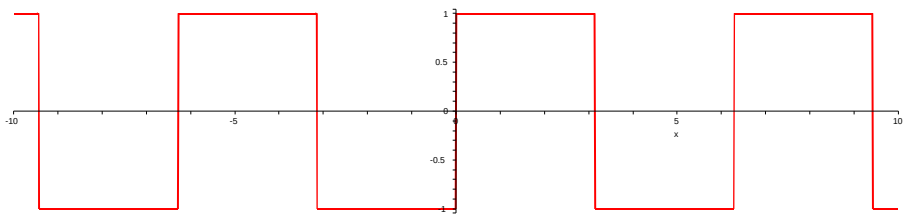
## An Example: The Square Wave

Let

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < 0 \\ 1 & \text{for} \quad 0 < x < \pi, \end{cases}$$

with $f(x + 2\pi) = f(x)$ for all $x$.

Square Wave




**Problem.** Find the Fourier series of $f(x)$.

**Solution.** Since $f(x)$ is odd, $a_n = 0$ for $n = 0, 1, 2, \ldots$.

Next,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(nx) f(x), dx = \frac{2}{\pi} \int_0^{\pi} \sin(nx) \cdot 1 \, dx =$$

$$\frac{-2}{n\pi} (\cos(n\pi) - \cos(0)) = \begin{cases} 0 & \text{for} \quad n \text{ even} \\ \frac{4}{n\pi} & \text{for} \quad n \text{ odd.} \end{cases}$$

Thus,

$$f(x) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin\left((2k-1)x\right) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots .$$

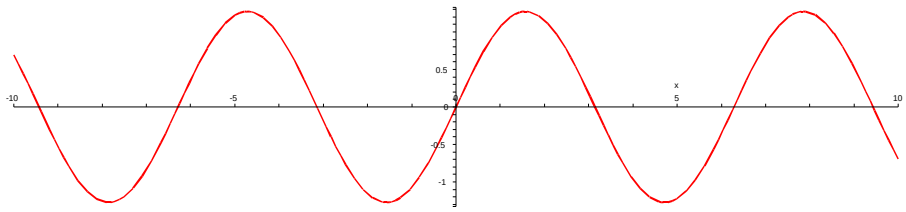
Recall that using $(2k - 1)$ is a way to generate only odd numbers.

On the next page we plot a few partial sums.

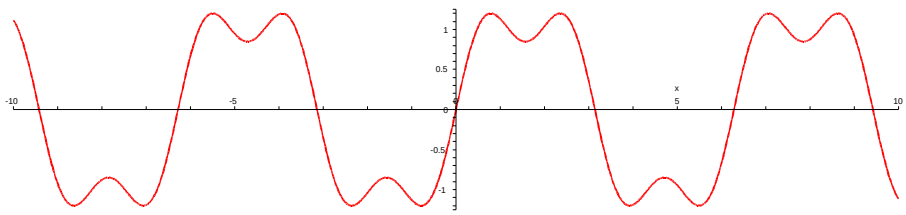
Let $f_N(x) = \sum_{k=1}^{N} \frac{4}{(2k-1)\pi} \sin\left((2k-1)x\right)$.

**Plots of $f_N(x)$ for $N = 1, 2, 3, 4, 10, 100$**

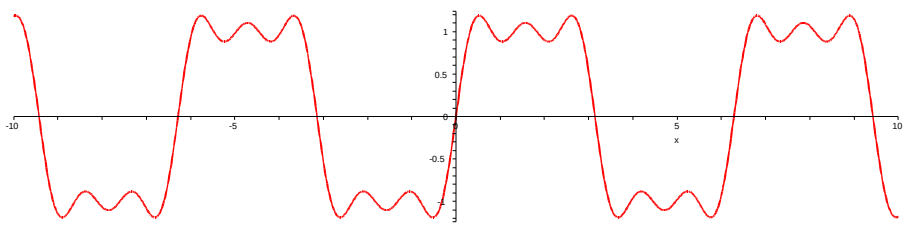
$N = 1$



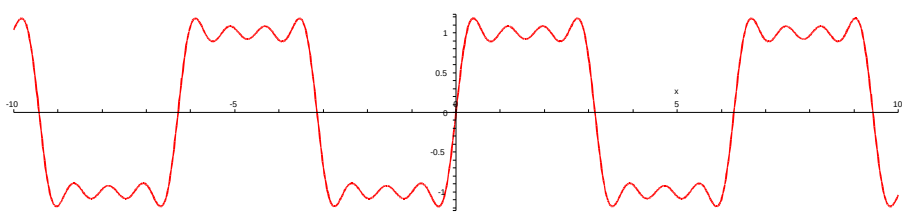

$N = 2$



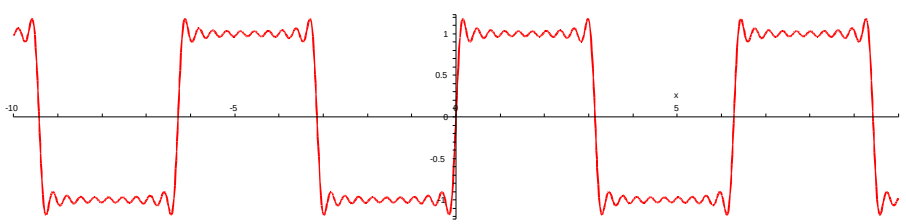

$N = 3$



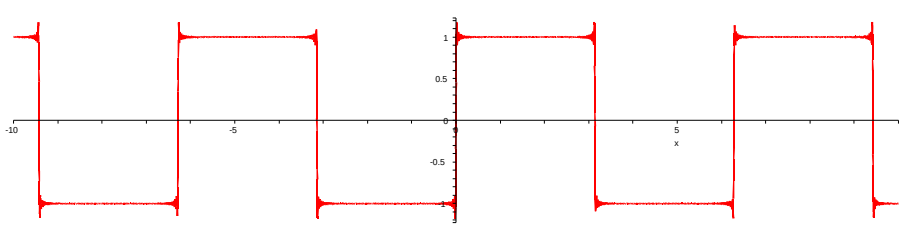

$N = 4$




$N = 10$

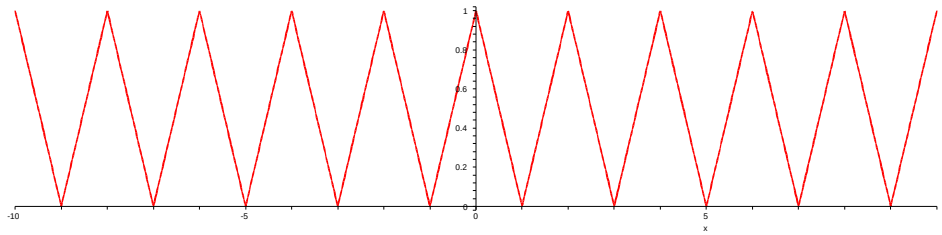



$N = 100$



**Another Example! A Triangle Wave**

Problem: Find the Fourier series of the wave depicted below.




Solution: First note that since the period is 2, $L = 1$. We can see that for $0 < x < 1$, $f(x) = 1-x$. Since this function is even, $b_n = 0$ for $n = 1, 2, 3, \ldots$. Also $\frac{a_0}{2}$ = ave. value of $f(x) = \frac{1}{2}$. Thus, $a_0 = 1$.

Now,

$$\begin{aligned}
a_n &= \frac{1}{1}\int_{-1}^{1} \cos(n\pi x) f(x)\, dx \\
&= 2\int_{0}^{1} \cos(n\pi x)(1-x)\, dx \\
&= \frac{2}{n\pi}\left(\sin(n\pi x) - x\sin(n\pi x) - \frac{1}{n\pi}\cos(n\pi x)\right)\Bigg|_0^1 \\
&= \frac{2}{n\pi}\left[\left(-\frac{1}{n\pi}\cos(n\pi)\right) - \left(-\frac{1}{n\pi}\right)\right] \\
&= \begin{cases} 0 & \text{for even } n \\ \frac{4}{n^2\pi^2} & \text{for odd } n. \end{cases}
\end{aligned}$$

Thus,

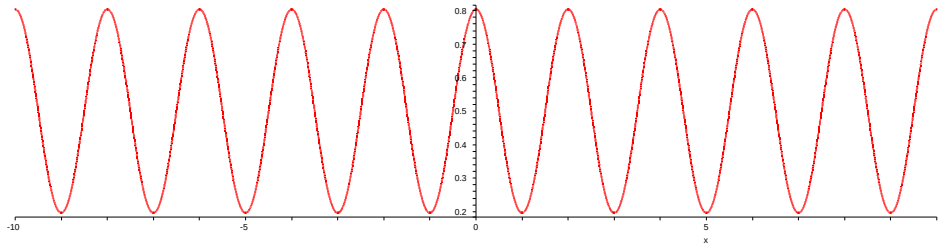
$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos\left((2k-1)\pi x\right).$$

On the next page we plot a few partial sums.

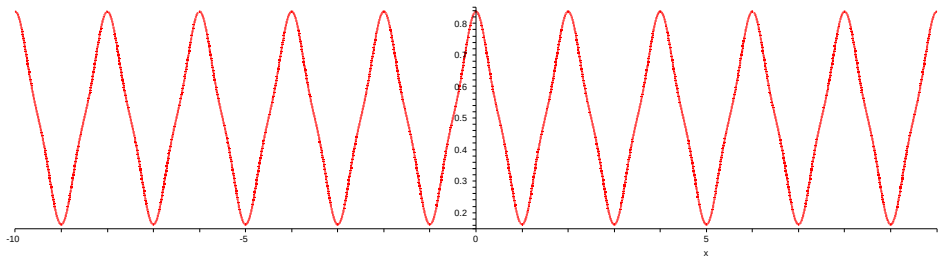
Let $f_N(x) = \frac{1}{2} + \sum_{k=1}^{N} \frac{4}{(2k-1)^2\pi^2} \cos\left((2k-1)\pi x\right)$.

**Plots of $f_N(x)$ for $N = 1, 2, 3, 30$**

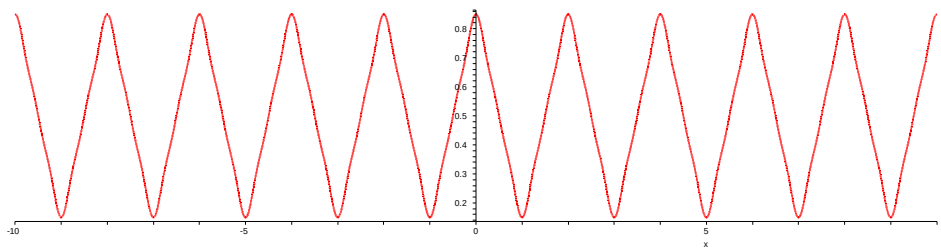
$N = 1$




$N = 2$




$N = 3$




$N = 30$