Exact First Order Differential Equations

This Lecture covers material in Section 2.6. A first order differential equations is exact if it can be written in the form

\[ M(x, y) + N(x, y) \frac{dy}{dx} = 0, \]

where

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \]

Before showing how to solve these we need to review some multi-variable calculus, especially the two-variable chain rule. This will also help to motivate why equations of this form are important in physics.

Let \( \psi(x, y) \) be a function of two variables. Then we can think of \( z = \psi(x, y) \) as a surface in three-dimensional space where \( z \) is the height above the \( xy \)-plane. Now suppose the \( x \) and \( y \) are functions of \( t \) (time) so that \( (x(t), y(t)) \) gives a curve in the \( xy \)-plane. Then \( z(t) = \psi(x(t), y(t)) \) gives a curve in three-dimensional space. Suppose we desire to know the rate of change of \( z \) with respect to \( t \). According to the two-variable chain rule the answer is

\[ \frac{dz}{dt} = \frac{\partial \psi}{\partial x} \frac{dx}{dt} + \frac{\partial \psi}{\partial y} \frac{dy}{dt}. \]

This formula is derived in Calculus III. Here I will give an intuitive motivation for why it works.

Suppose \( \psi(x, y) \) is just a plane. Then we have

\[ z = \psi(x, y) = Ax + By + C \]

for some constants \( A, B \) and \( C \). Here \( A \) is the slope of the plane with respect to the \( x \) direction, \( B \) is the slope of the plane with respect to the \( y \) direction and \( C \) is the intercept with the \( z \)-axis.
We want to compute the change in $z$ as $t$ changes from $t_0$ to $t_0 + \Delta t$. Let $\Delta x = x(t_0 + \Delta t) - x(t_0)$ and $\Delta y = y(t_0 + \Delta t) - y(t_0)$ be the changes in $x$ and $y$, respectively. For convenience let $x_0 = x(t_0)$ and $y_0 = y(t_0)$. Then the change in $z$ is

$$
\Delta z = \psi(x_0 + \Delta x, y_0 + \Delta y) - \psi(x_0, y_0) = A\Delta x + B\Delta y.
$$

We divide both sides by $\Delta t$ to obtain

$$
\frac{\Delta z}{\Delta t} = A\frac{\Delta x}{\Delta t} + B\frac{\Delta y}{\Delta t}.
$$

Now we can find the derivative of $z$ with respect to $t$ by taking limits as $\Delta t \to 0$. This gives

$$
\frac{dz}{dt} = \frac{A}{dt} + B\frac{dy}{dt}.
$$

But notice that $A = \partial_x \psi$ and $B = \partial_y \psi$. Thus we have

$$
\frac{dz}{dt} = \frac{\partial \psi}{\partial x} \frac{dx}{dt} + \frac{\partial \psi}{\partial y} \frac{dy}{dt}
$$

which is (\*).

This shows that the two-variable chain rule works for planes. In general if $\psi(x, y)$ is reasonably smooth it can be approximated near each point by a tangent plane. It can be shown that this gives the two-variable chain rule for any function of two variables that
is smooth enough that its graph has a tangent plane at each point in an open set containing the point of interest.

Now, let \( z = \psi(x, y) \) be a surface. But suppose \( z \) is some quantity that is conserved, like energy. That is we now have

\[
\psi(x, y) = C.
\]

The slice of the surface through \( z = C \) is called **level curve**.

Example: Let \( z = \psi(x, y) = x^2 + y^2 \). Then the level curve for \( z = 1 \) is a circle of radius 1 that floats one unit above the \( xy \)-plane.

Example: Let \( z = \psi(x, y) = 3x + y - 3 \). The level curve for \( z = 2 \) is the line \( 3x + y - 3 = 2 \), or \( y = -3x + 5 \), that is it floating two units above the \( xy \)-plane.

For now suppose \( y \) is a function of \( x \) (at least implicitly). As we change \( x \) we cause \( y \) to change so that \( z = \psi(x, y) \) stays on the same level curve. Since \( z \) is not changing we have \( dz/dx = 0 \). The two-variable chain rule, using \( x \) for \( t \), gives

\[
0 = \frac{dz}{dx} = \frac{\psi(x, y(x))}{dx} = \frac{\partial \psi}{\partial x} \frac{dx}{dx} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}.
\]

Therefore,

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} y' = 0.
\]

If \( \psi_x \) and \( \psi_y \) are known functions what we have is a differential equation in \( y \).

We will be doing the inverse of this process. That is, given at differential equation in the form

\[
M(x, y) + N(x, y) \frac{dy}{dx} = 0
\]

we will solve it for \( y(x) \) (or at least a relation between \( x \) and \( y \)) by finding a surface \( \psi(x, y) \) such that \( M = \psi_x \) and \( N = \psi_y \), and then using an initial condition to find the desired level curve. In many applications \( \langle M, N \rangle \) is given as a force field and then \( \psi \) is a
potential energy function. If energy is conserved, the dynamics are restricted to a level curve of \( z = \psi(x, y) \).

Enough talk, let’s do some examples.

**Example 1.** Solve \((2x + y) + (x + 2y)y' = 0\), with \(y(3) = 1\).

**Solution.** We want to find a function \( \psi(x, y) \) such that
\[
\frac{\partial \psi}{\partial x} = 2x + y \quad \& \quad \frac{\partial \psi}{\partial y} = x + 2y.
\]

So, we integrate.
\[
\psi = \int \psi_x \, dx = \int 2x + y \, dx = x^2 + xy + C_1(y),
\]

where \( C_1(y) \) an arbitrary function of \( y \). The idea is we are finding the class of all functions whose partial derivative with respect to \( x \) gives \( 2x + y \).

But we also need for \( \psi_y = x + 2y \). So, we integrate.
\[
\psi = \int \psi_y \, dy = \int x + 2y \, dy = xy + y^2 + C_2(x),
\]

where \( C_2(x) \) can be any function of \( x \).

We now have two classes of functions, each satisfying one of the two conditions. If we could find a function that is in both class that would do the trick. The answer is obvious. Let
\[
\psi(x, y) = x^2 + xy + y^2.
\]

This function is in both classes and thus satisfies both the needed conditions. Now we consider the initial condition, \( y(3) = 1 \), that is, \( x = 3 \implies y = 1 \). Then
\[
\psi(3, 1) = 9 + 3 + 1 = 13.
\]

Thus, the level curve we want is
\[
x^2 + xy + y^2 = 13.
\]
We will leave as a relation. Below are plots of the surface $z = x^2 + xy + y^2$ with the level 13 curve and a projection of this curve into the $xy$-plane.

**Extra Credit.** Prove that this curve $x^2 + xy + y^2 = 13$ is an ellipse and find its focal points. You can do this by reviewing how to rotate graphs with rotation matrices and the properties of ellipses. Then rotate the graph $45^\circ$ so that its major axis lies along the x-axis.

**Example 2 (Not!).** Solve $(2x + 2y) + (x + 2y)y' = 0$, with $y(3) = 1$. We integrate.

\[
\psi = \int 2x + 2y \, dx = x^2 + 2xy + C_1(y).
\]

\[
\psi = \int x + 2y \, dy = xy + y^2 + C_2(x).
\]

Now look closely. Since $2xy \neq xy$ there is no function that meets both conditions. The method fails! What this means in physical terms is that the force field $\langle 2x + 2y, x + 2y \rangle$ does not arise from a potential function; in such a system energy is **not conserved**.

What we need is a quick test to see if $\psi$ exists for a given equation so that we don’t waste a lot of time barking up the wrong tree.

**Theorem!** Given two functions $M(x, y)$ and $N(x, y)$, there exists a function $\psi(x, y)$ such that

\[
\frac{\partial \psi}{\partial x} = M(x, y) \quad \& \quad \frac{\partial \psi}{\partial y} = N(x, y),
\]
if and only if
\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \]
in an open rectangle containing the point of interest.

Check this for the two examples above. This is Theorem 2.6.1 in your textbook. Your textbook gives a proof, but another prove is covered in Calculus III that uses Green’s Theorem. If you are a Math major read both and compare them. However, one direction is easy: if \( \psi \) exists, then \( \psi_{xy} = \psi_{yx} \implies M_y = N_x \). This theorem is the motivation for the definition we gave at the beginning of an exact first order differential equation.

**Example 3.** Find the general solution to
\[ y \cos x + ye^{xy} + (\sin x + xe^{xy})y' = 0. \]

*Solution.* Let \( M = y \cos x + ye^{xy} \) and \( N = \sin x + xe^{xy} \). Then
\[ M_y = \cos x + e^{xy} + xye^{xy} = N_x. \]
Thus, it is exact. We integrate.

\[ \psi = \int M \, dx = y \sin x + e^{xy} + C_1(y) \]
and
\[ \psi = \int N \, dy = y \sin x + e^{xy} + C_2(x). \]

We let \( \psi(x, y) = y \sin x + e^{xy} \). The general solution is then
\[ y \sin x + e^{xy} = C. \]

**Example 4.** Solve \( 4x^3 + 4y^3y' = 0 \), with \( y(1) = 1 \).

*Solution.* It is exact since \((4x^3)_y = 0\) and \((4y^3)_x = 0\). Then
\[ \psi = \int 4x^3 \, dx = x^4 + C_1(y) \]
and

\[ \psi = \int 4y^3 \, dy = y^4 + C_2(x). \]

We let \( \psi = x^4 + y^4 \). Since (1,1) is our initial condition we see that our solution is

\[ x^4 + y^4 = 2. \]

Below is a graph of this curve projected into the \( xy \)-plane.

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**Example 5.** Solve \( 4x^4 + 4xy^3y' = 0 \), with \( y(1) = 1 \).

**Solution.** We check for exactness. \( (4x^4)_y = 0 \) while \( (4xy^3)_x = 4y^3 \). Thus it is not exact. But wait! Notice that this example is exactly the same as Example 4, but that we have multiplied through by \( x \). So, if we now multiple through by \( \frac{1}{x} \) we get

\[ 4x^3 + 4y^3y' = 0. \]

Thus, the solution is same as in Example 4!

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**Integrating factors.**

This last example motivates the following idea. Suppose we have a differential equation of the form

\[ M + Ny' = 0 \]

which is not exact. Can we find a function \( \mu(x, y) \) such that

\[ \mu M + \mu Ny' = 0 \]
is exact?

The answer is, not always, but sometimes you can. When this works we call $\mu$ an **integrating factor**. Finding such as $\mu$ can be tricky. Here we show three special cases where an integrating factor $\mu$ can be found. Each relies on an assumption about $\mu$ that can be tested for.

**Case 1.** Suppose a suitable $\mu$ exists and that it is a function of $x$ only.

**Case 2.** Suppose a suitable $\mu$ exists and that it is a function of $y$ only.

**Case 3.** Suppose a suitable $\mu$ exists and that it can be written as a function dependent only on the product $xy$.

In all cases we need to find $\mu$ such that

$$(\mu M)_y = (\mu N)_x$$

so that we have exactness. By the product rule this is equivalent to requiring

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x. \quad (*)$$

**Case 1.** If Case 1 holds then $\mu_y = 0$ and we can think of $\mu_x$ as $\mu'$. Then $(*)$ becomes

$$\mu M_y = \mu' N + \mu N_x$$

or

$$\frac{\mu'}{\mu} = \frac{M_y - N_x}{N}.$$ 

If our assumption is correct then, since $\mu'/\mu$ depends only on $x$, we know that $(M_y - N_x)/N$ depends only on $x$. Then the integrals
below are well defined.
\[
\int \frac{1}{\mu} \, d\mu = \int \frac{M_y - N_x}{N} \, dx
\]
Thus,
\[
\mu = e^{\int \frac{M_y - N_x}{N} \, dx}.
\]
In fact, this gives us a test to determine when this method will work. If \( \frac{M_y - N_x}{N} \) depends only on \( x \) it follows that \( \mu \) depends only on \( x \).

**Example 6.** Find the general solution to \( y^2 + x^3 + xy y' = 0 \).

*Solution.* Since \( (y^2 + x^3)_y = 2y \) and \( (xy)_x = y \) are not equal, this equation is not exact. But
\[
\frac{2y - y}{xy} = \frac{1}{x}
\]
depends only on \( x \). Thus we let
\[
\mu = e^{\int \frac{1}{x} \, dx} = x.
\]
So, we multiply through by \( x \) to get
\[
x y^2 + x^4 + x^2 y y' = 0.
\]
Let \( M = xy^2 + x^4 \) and \( N = x^2 y \). Then \( M_y = 2xy = N_x \), so we have exactness. Now we find \( \psi \) as before.
\[
\psi = \int M \, dx = \frac{1}{2} x^2 y^2 + \frac{1}{5} x^5 + C_1(y)
\]
\[
\psi = \int N \, dy = \frac{1}{2} x^2 y^2 + C_2(x)
\]
Thus, we let \( \psi = \frac{1}{2} x^2 y^2 + \frac{1}{5} x^5 \), so the general solution is
\[
\frac{1}{2} x^2 y^2 + \frac{1}{5} x^5 = C,
\]
or if you prefer
\[
5 x^2 y^2 + 2 x^5 = C.
\]
Solving for $y$ gives
\[ y = \pm \sqrt{\frac{C - 2x^5}{5x^2}}. \]

**Case 2.** This is so similar to Case 1 that we leave it to you to develop the method and find the formula for $\mu(y)$.

**Case 3.** Recall equation $(\ast)$: $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. Let $v = xy$ and remember we are assuming $\mu$ can be rewritten as a function of $v$. Thus,
\[ \mu_y = \frac{\partial \mu(v)}{\partial y} = \frac{d\mu}{dv} \frac{\partial v}{\partial y} = \frac{d\mu}{dv} \cdot x = x\mu', \]
and
\[ \mu_x = \frac{\partial \mu(v)}{\partial x} = \frac{d\mu}{dv} \frac{\partial v}{\partial y} = \frac{d\mu}{dv} \cdot y = y\mu', \]
where $\mu'$ means the derivative with respect to $v$. Now $(\ast)$ becomes
\[ x\mu' M + \mu M_y = y\mu' N + \mu N_x, \]
which gives
\[ \frac{\mu'}{\mu} = \frac{N_x - M_y}{xM - yN}. \]
If the right hand side depends only on $v = xy$ then the assumption we are making is valid, and thus
\[ \mu = e^{\int \frac{N_x - M_y}{xM - yN} dv}. \]
Perhaps an example would help.

**Example 7.** Solve
\[ 5x^3 + \frac{1}{x} \cos xy + \frac{x^4 + \cos xy \, dy}{y} \frac{dy}{dx} = 0, \]
with \( y(1) = \pi \).

**Solution.** Let \( M = 5x^3 + \frac{1}{x} \cos xy \) and \( N = \frac{x^4 + \cos xy}{y} \). Then

\[
M_y = -\sin xy \quad \& \quad N_x = \frac{4x^3 - y \sin xy}{y}
\]

Thus, the given equation is not exact. We now search for an integration factor.

**Case 1.** \( \frac{M_y - N_x}{N} = \frac{-4x^3}{y} = \frac{-4x^3}{x^4 + \cos xy} \) No good!

**Case 2.** \( \frac{N_x - M_y}{M} = \frac{4x^3 y}{5x^4 + \cos xy} = \frac{4x^4}{5x^4 + y \cos xy} \) Rats!!

**Case 3.** \( \frac{N_x - M_y}{xM - yN} = \frac{4x^3}{5x^4 + \cos xy - (x^4 + \cos xy)} = \frac{1}{xy} ! \) Eureka!!

Let \( v = xy \). Now,

\[
\mu(v) = e^{\int \frac{1}{v} dv} = e^{\ln|v| + C} = C|v| = C|xy|;
\]

we will use \( \mu = xy \)

On ward! We multiply the original equation by \( xy \) to get

\[
5x^4y + y \cos xy + (x^5 + x \cos xy)y' = 0.
\]

Let \( M = 5x^4y + y \cos xy \) and \( N = x^5 + x \cos xy \). We double check that it is in fact exact.

\[
M_y = 5x^4 + \cos xy - xy \sin xy = N_x.
\]

Now the hunt is on for \( \psi \! \\
\]

\[
\psi = \int M \, dx = x^5y + \sin xy + C_1(y)
\]
and

$$\psi = \int N \, dy = x^5 y + \sin xy + C_2(x).$$

Thus, $\psi = x^5 y + \sin xy$ and the general solution is $x^5 y + \sin xy = C$. Since $y(1) = \pi$ you can check that $C = \pi$. Thus, the solution is

$$x^5 y + \sin xy = \pi.$$