Second Order Differential Equations that can be Transformed into First Order Differential Equations

This Lecture covers material developed in the exercises (36-51) on pages 135–136 of the Boyce & DiPrima textbook.

A second order differential equation is one that involves second derivatives. Normally they have two initial conditions.

**Example 1.** Consider \( y'' + \frac{y'}{t + 1} = 2 \) with \( y(0) = 1 \) and \( y'(0) = 2 \).

**Solution.** Notice that there is no \( y \) term. If we let \( v = y' \) we get

\[
v' + \frac{v}{t + 1} = 2 \text{ with } v(0) = 2.
\]

Now this is a first order equation. It is linear. Let,

\[
\mu(t) = e^{\int \frac{1}{t+1} dt} = e^{\ln |t+1| + C} = C|t+1|.
\]

We will use \( \mu = t + 1 \). Now we have,

\[
(t + 1)v' + v = 2(t + 1)
\]

\[
((t + 1)v)' = 2t + 2
\]

\[
(t + 1)v = t^2 + 2t + C_1
\]

\[
v = \frac{t^2 + 2t + C_1}{t + 1}
\]

Since \( v(0) = 2 \) we get \( C_1 = 2 \). Now we find \( y(t) \).

\[
y' = v
\]

\[
y = \int \frac{t^2 + 2t + 2}{t + 1} dt
\]

\[
y = \int t + 1 + \frac{1}{t + 1} dt, \quad \text{(by long division)}
\]

\[
y = \frac{1}{2}t^2 + t + \ln |t + 1| + C_2.
\]
Since \( y(0) = 1 \) have \( 1 = 0 + 0 + \ln |1| + C_2 \). Hence \( C_2 = 1 \). Finally,
\[
y(t) = \frac{1}{2} t^2 + t + \ln(t + 1) + 1, \quad \text{for } t > -1.
\]

\[\square\]

**Example 2.** Solve \( y'y'' = 2 \), with \( y(0) = 1 \) and \( y'(0) = 2 \).

*Solution.* Again \( y \) does not appear. Let \( v = y' \). Then we get \( vv' = 2 \). This is separable.

\[
\int v \, dv = \int 2 \, dt.
\]
\[
\frac{1}{2} v^2 = 2t + C_1.
\]
\[
\frac{1}{2} v^2 = 2t + 2, \quad \text{since } v(0) = y'(0) = 2.
\]
\[
v = \pm \sqrt{4t + 4}.
\]
\[
v = \sqrt{4t + 4}, \quad \text{since } v(0) = 2 > 0.
\]

Now integrate \( v \) to get \( y \).

\[
y = \int v \, dt
\]
\[
= 2 \int \sqrt{t + 1} \, dt
\]
\[
= \frac{4}{3} (t + 1)^{\frac{3}{2}} + C_2
\]
\[
= \frac{4}{3} (t + 1)^{\frac{3}{2}} - \frac{1}{3}, \quad \text{since } y(0) = 1.
\]

Thus,
\[
y(t) = \frac{4(t + 1)^{\frac{3}{2}} - 1}{3} \quad \text{for } t > -1.
\]

\[\square\]
Example 3. Find the general solution to \( yy'' + (y')^2 = 0 \), with independent variable \( t \). Notice that \( t \) does not appear, except in the differentiation symbols \( \frac{d}{dt} \) if we write it out "longhand".

Solution. It turns out the same substitution \( v = y' \) will work, but the steps are different. The result at first looks innocent enough:

\[ yv' + v^2 = 0. \]

It looks separable, but this is not valid. The reason is clearer if we write it as

\[ y \frac{dv}{dt} + v^2 = 0. \]

There are three variables. The \( v' \) did not mean \( \frac{dv}{dy} \) so this not an equation form we have studied.

The way around this is to think of \( v \) as a function of \( y \). Then

\[ \frac{d}{dt} v(y(t)) = \frac{dv}{dy} \frac{dy}{dt} = \frac{dv}{dy} v = v'v, \]

where \( v' \) is understood to mean the derivative with respect to \( y \). Now we have

\[ yvv' + v^2 = 0. \]

This is now a true first order equation. Dividing by \( v \) makes in linear.

\[
\begin{align*}
yv' + v &= 0 \\
(yv)' &= 0 \\
yv &= C_1 \\
v &= C_1/y
\end{align*}
\]
Now convert back to $y$ and $t$.

\[
\frac{dy}{dt} = \frac{C_1}{y}
\]

\[
\int y \, dy = \int C_1 \, dt
\]

\[
\frac{1}{2}y^2 = C_1 t + C_2
\]

\[
y = \pm \sqrt{C_1 t + C_2}
\]

This then is the general solution. If we had a pair of initial conditions we could find $C_1$ and $C_2$ and resolve the ± sign. □
**Example 4.** Find the general solution to \( y'' + y(y')^3 = 0 \). Let \( t \) be the independent variable.

**Solution.** Let \( v = y' \). Then use

\[
\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v'v,
\]

where the last \( v' \) now means the derivative with respect to \( y \). Then the problem becomes,

\[
v v' + yv^3 = 0,
\]

or

\[
v' = -yv^2,
\]

which is separable. We have

\[
\int v^{-2} \, dv = \int -y \, dy
\]

\[
-v^{-1} = -\frac{1}{2}y^2 + C_1
\]

\[
v = \frac{1}{\frac{1}{2}y^2 - C_1}
\]

\[
\frac{dy}{dt} = \frac{2}{y^2 + C_1}, \quad (\text{new } C_1)
\]

\[
\int y^2 + C_1 \, dy = \int 2 \, dt
\]

\[
\frac{1}{3}y^3 + C_1y + C_2 = 2t
\]

\[
y^3 + C_1y + C_2 = 6t \quad (\text{new } C_1\&C_2)
\]

There is no simple way to solve for \( y \), so we’ll leave it in this form. But, notice that \( y = C \) also works for certain initial conditions. For example, suppose \( y(0) = 2 \) and \( y'(0) = 0 \). Then \( y(t) = 2 \) is a
solution. Further, the solution we found will fail! Notice $y'(0) = 0$ means
\[ y'(0) = \frac{2}{y^2 + C_1} = \frac{2}{4 + C_1} = 0. \]
But there is no value of $C_1$ that works!

Now try $y(0) = 0$ and $y'(0) = 1$. This time $y(t)$ equal to a constant won’t work since its derivative could never be 1. But,

\[ y(0) = 0 \implies C_2 = 0 \]
and
\[ y'(0) = 1 \implies \frac{2}{1 + C_1} = 1 \implies C_1 = 1. \]
Thus, the solution is
\[ y^3 + y = 6t. \]
In this case the function $y^3 + y$ is one-to-one so we could graphically find $y$ for a given value of $t$. \[ \Box \]

Extra Credit. Suppose the initial conditions in Example 4 are $y(0) = a$ and $y(0) = b$. (So, no $y'(0)$.) Show that there are unique values for $C_1$ and $C_2$, unless $a = b$. What is the solution when $a = b$?
Example 5. Solve $y'' + (y')^2 - 4y = 2$, with $y(0) = 0$ and $y'(0) = 0$. Let $t$ be the independent variable.

Solution. Let $v = y' = \frac{dy}{dt}$. Then use

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v'v,$$

where the last $v'$ now means the derivative with respect to $y$. Then the problem becomes,

$$vv' + v^2 - 4y = 2,$$

with $y$ taken as the independent variable. Rewrite as

$$(v^2 - 4y - 2) + vv' = 0.$$ 

Let $M = v^2 - 4y - 2$ and $N = v$. Then

$$\frac{\partial M}{\partial v} = 2v \quad \frac{\partial N}{\partial y} = 0.$$ 

It is not exact, but

$$\frac{M_v - N_y}{N} = 2,$$ 

which does not depend on $v$. Thus, we use $\mu = e^{2y}$ as an integrating factor. Now we have

$$e^{2y}(v^2 - 4y - 2) + e^{2y}vv' = 0.$$ 

You can check that it is exact. Now,

$$\psi(y,v) = \int (v^2 - 4y - 2)e^{2y} \, dy = \frac{1}{2}(v^2 - 2)e^{2y} + (1 - 2y)e^{2y} + C_1(v)$$

$$= \frac{1}{2}v^2e^{2y} - 2ye^{2y} + C_1(v),$$

and

$$\psi(y,v) = \int ve^{2y} \, dv = \frac{1}{2}v^2e^{2y} + C_2(y).$$
If we let $C_1(v) = 0$ and $C_2(y) = -2ye^{2y}$ we have our solution:

$$\psi(y, v) = \frac{1}{2}v^2e^{2y} - 2ye^{2y} = C_3.$$ 

At $t = 0$ both $y(0)$ and $v(0) = y'(0) = 0$. Thus $C_3$ must be zero. Now we can simplify and get

$$v^2 = 4y \quad \text{or} \quad (y')^2 = 4y.$$ 

Thus, $y' = \pm 2\sqrt{y}$, which is separable. Next

$$\int y^{-1/2} \, dy = \pm \int 2 \, dt = \pm 2t + C_4.$$ 

Thus, $2y^{1/2} = \pm 2t + C_4$. Since $y(0) = 0$ we get that $C_4 = 0$. Hence, the solution is

$$y(t) = t^2.$$

□
Summary

In this lecture we have developed two methods for reducing certain second order differential equations to first order differential equations. Both start with a substitution or “change of variables” given by

\[ v = \frac{dy}{dt}. \]

**Case 1.** If the given equation does not contain any “y” terms, replace all occurrences of \( y' \) with \( v \) and of \( y'' \) with \( v' \). In this case the derivatives are all with respect to the original independent variable, \( t \). Once you solve for \( v(t) \) integrate it to get \( y(t) \). The final result will have two arbitrary constants. This was the method used in Examples 1 & 2.

**Case 2.** If the given equation does not contain any “t” terms, the situation is trickier. You replace any occurrences of \( y' \) with \( v \), but for \( y'' \) you use the following:

\[\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v'v,\]

where \( v' \) means the derivative with respect to \( y \). The resulting equation is a first order differential equation in \( v \) with independent variable \( y \).

Solve it for \( v(y) \). Then you have a problem of the form

\[ \frac{dy}{dt} = v(y) \]

which is separable, in fact it is autonomous. Solve it by

\[ \int \frac{1}{v(y)} \, dy = \int dt = t + C_2, \]

then solve the result for \( y(t) \) if possible.

Finally, check by inspection to see if \( y(t) = \) a constant will work. Solutions may not work for all initial conditions.

This was the method used in Examples 3, 4, & 5.