Homogeneous Second Order Differential Equations with Constant Coefficients: Continued

Complex roots.

The final case is what to do when the roots of the characteristic polynomial are complex. Recall that this means they will be of the form \( p \pm qi \) for real numbers \( p \) and \( q \) where \( i^2 = -1 \). (This assumes that the coefficients \( a \), \( b \) and \( c \) are real.) We will need to “review” some facts about complex functions that were censored from your calculus textbook.

But first, let’s look at a simple example, \( y'' + y = 0 \). We can rewrite this as \( y'' = -y \). So, we are seeking functions whose second derivatives are their own negatives. Two might come to mind, \( \sin x \) and \( \cos x \). In fact \( y = C_1 \sin x + C_2 \cos x \) gives all possible solutions, as we will show later.

The roots of the characteristic polynomial, \( r^2 + 1 = 0 \), are \( \pm i \). Notice that
\[
(e^{ix})'' = (ie^{ix})' = i^2 e^{ix} = -e^{ix}.
\]
But, what does it mean to raise \( e \) to a complex power? And, what does it mean to take a derivative of such a function? And, how are these functions connected to \( \sin x \) and \( \cos x \)?

Way back in Calculus II you studied Taylor series and you learned that
\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.
\]
Suppose \( z = a + ib \) is a complex number. Then we define
\[
e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots.
\]
In courses on Complex Analysis (MATH 455 here) it is shown that this sequence converges for all complex numbers \( z \). The derivative can be defined via term-by-term differentiation. The following facts can also be proven:
\[
e^{a+ib} = e^a e^{ib}
\]
\[ \frac{de^{\alpha ix}}{dx} = \alpha ie^{\alpha ix}, \]

for any real (or complex) number \( \alpha \). Now watch.

\[ e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \cdots \]

\[ = 1 + ix - \frac{x^2}{2!} - \frac{i x^3}{3!} + \frac{x^4}{4!} + \frac{i x^5}{5!} - \frac{x^6}{6!} - \frac{i x^7}{7!} + \frac{x^8}{8!} + \cdots \]

\[ = \left( 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \right) + i \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right) \]

\[ = \cos x + i \sin x. \]

Now let’s get back to differential equations. Suppose we have \( ay'' + by' + cy = 0 \) and the roots of \( ar^2 + br + c \) are \( \alpha \pm i\beta \). Then the general solution is

\[ y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} = (C_1 e^{i\beta x} + C_2 e^{-i\beta x}) e^{\alpha x} \]

\[ = (C_1 \cos(\beta x) + i \sin(\beta x)) + C_2 (\cos(\beta x) - i \sin(\beta x))) e^{\alpha x} \]

\[ = ((C_1 + C_2) \cos(\beta x) + i (C_1 - C_2) \sin(\beta x)) e^{\alpha x}. \]

We can rewrite this as

\[ Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x. \]

**Theorem 3.** The general solution to \( ay'' + by' + cy = 0 \) when the roots of the characteristic polynomial are \( \alpha \pm i\beta \) is

\[ y = Ae^{\alpha x} \cos \beta x + Be^{\alpha x} \sin \beta x. \]

If \( y(x_0) = p \) and \( y'(x_0) = q \) then there is a unique solution for \( A \) and \( B \).

**Proof.** We have already derived the solution, but you can check it by directly substituting it in to the differential equation. Next, \( y(x_0) = p \) implies

\[ A \cos \beta x_0 + B \sin \beta x_0 = pe^{-\alpha x_0}. \]
And \( y'(x_0) = q \) implies
\[ A\alpha e^{\alpha x_0} \cos \beta x_0 - A\beta e^{\alpha x_0} \sin \beta x_0 + B\alpha e^{\alpha x_0} \sin \beta x_0 + B\beta e^{\alpha x_0} \cos \beta x_0 = q, \]
or
\[ A(\alpha \cos \beta x_0 - \beta \sin \beta x_0) + B(\beta \cos \beta x_0 + \alpha \sin \beta x_0) = qe^{-\alpha x_0}. \]
So, again we have two equations and two unknowns and these can readily be solved for \( A \) and \( B \).

**Example.** Find the general solution to \( y'' - y' + 2y = 0 \). Then find the solution for the initial values \( y(0) = p, \ y'(0) = q \).

**Solution.** The characteristic polynomial \( r^2 - r + 2 \) has complex roots \( r = \frac{1}{2} \pm i\frac{\sqrt{7}}{2} \). Thus, the general solution is
\[ y(x) = Ae^{\frac{1}{2}x} \cos \frac{\sqrt{7}}{2}x + Be^{\frac{1}{2}x} \sin \frac{\sqrt{7}}{2}x. \]

Now, \( y(0) = p \) implies \( A = p \) and \( y'(0) = q \) gives \( p/2 + B\sqrt{7}/2 = q \). Thus, \( B = \frac{2q-p}{\sqrt{7}} \) and we have
\[ y(x) = pe^{\frac{1}{2}x} \cos \frac{\sqrt{7}}{2}x + \frac{2q-p}{\sqrt{7}}e^{\frac{1}{2}x} \sin \frac{\sqrt{7}}{2}x. \]