Ch 6

Laplace Transforms

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The Laplace Transform provides a method for solving differential equations that will be most helpful to us for diff eqs of the form:

$$ay'' + by' + y = g(t), \quad y'(0) = \alpha, \quad y(0) = \beta,$$

where $g(t)$ has discontinuities. But it will take some effort to develop this tool before we see how to use it.

Def The Laplace Transform of a function $f(t)$ is a new function $F(s)$ given by

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

Note: the domain of $F(s)$ is often restricted to a interval of the form (a, ∞).

Ex Let $f(t) = 1$. Then $L(f) = \int_0^\infty e^{-st}\, dt =$

$$-\frac{1}{s} e^{-st} \Big|_0^\infty = 0 - \frac{-1}{s} e^0 = \frac{1}{s}, \text{ provided } s > 0.$$

Ex Let $f(t) = \sin t$. Then $L(f) = \int_0^\infty e^{-st} \sin t\, dt =$

$$-\frac{1}{s^2+1} e^{-st} (\cos t + s\sin t) \Big|_0^\infty = 0 - \frac{-1}{s^2+1} e^0 (1 + s \cdot 0) =$$

$$\frac{1}{s^2+1}, \text{ provided } s > 0.$$

Ex Let $f(t) = e^{ct}$. Then $L(f) = \int_0^\infty e^{-st} e^{ct}\, dt$

$$= \int_0^\infty e^{(c-s)t}\, dt = \frac{1}{c-s} e^{(c-s)t} \Big|_0^\infty =$$

$$0 - \frac{1}{c-s} e^0 = \frac{1}{s-c}, \text{ provided } s > c.$$

Thm (6.1.2) Suppose
1. f is piecewise continuous on $[0, \infty)$, and
2. f is eventually dominated by an exponential function, i.e. $\exists\, a \in \mathbb{R}$, $K, M > 0$ with

$$|f(t)| \le K e^{at} \quad \forall t \ge M.$$

Then $L(f) = F(s)$ exists for $s > a$.

Proof See text.

Thm (6.2.1) For suitable $f(t)$, (see text)

$$\text{(A)} \quad L(f') = s L(f) - f(0)$$

and

$$\text{(B)} \quad L(f'') = s^2 L(f) - s f(0) - f'(0).$$

Pf (A) $L(f') = \int_0^\infty e^{-st} f'(t)\, dt = e^{-st} f(t) \Big|_0^\infty - -s \int_0^\infty e^{-st} f(t)\, dt$

$$= 0 - e^0 f(0) + s L(f) = s L(f) - f(0)$$

(B) repeat.

Ex Solve $y'' + y = 0$ $y'(0)=1$ $y(0)=0$.

$$L(y'') = s^2 L(y) - s\,y(0) - y'(0)$$
$$L(0) = 0.$$

$$L(y''+y) = L(y'') + L(y)$$

$$= s^2 L(y) - 1 + L(y) = 0$$

Solve for $L(y)$: $(s^2+1)L(y) = 1$

so $L(y) = \dfrac{1}{s^2+1}$

Thus $y = L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$.

Question: What happens if $y'(0)=0$, $y(0)=1$?

Introduce Table 6.2.1.

Ex #2 Given $F(s) = \dfrac{4}{(s-1)^3}$, find $f(t) = L^{-1}(F)$.

By Table item #11, $L(t^2 e^t) = \dfrac{2}{(s-1)^3}$.

So, $L^{-1}(F) = 2t^2 e^t$.

Ex #6 Given $F(s) = \dfrac{2s-3}{s^2-4}$, find $f(t) = L^{-1}(F)$.

$$\frac{2s-3}{s^2-4} = \frac{2s-3}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}$$

Need $2s-3 = As + A2 + Bs - B2 = (A+B)s + 2(A-B)$.

$$A+B=2$$
$$A-B=-\tfrac{3}{2}$$

$2A = \frac{1}{2}$ $A = \frac{1}{4}$ and $B = \frac{7}{4}$.

Thus, $L^{-1}(F(s)) = L^{-1}\left(\dfrac{2s-3}{s^2-4}\right) = L^{-1}\left(\dfrac{1/4}{s-2} + \dfrac{7/4}{s+2}\right)$

$= \frac{1}{4} L^{-1}\left(\frac{1}{s-2}\right) + \frac{7}{4} L^{-1}\left(\frac{1}{s+2}\right) = \frac{1}{4} e^{2t} + \frac{7}{4} e^{-2t}$

use Table item

Ex #12 Solve $y'' + 3y' + 2y = 0$ $y(0) = 1$, $y'(0) = 0$.

1. Apple L. $s^2 L(y) - s\cdot 1 - 0 + 3sL(y) - 3\cdot 1 + 2\cdot L(y) = L(0) = 0$
2. Solve for $L(y)$ $(s^2+3s+2)L(y) = s+3$

$$L(y) = \frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}$$

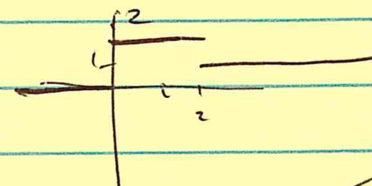
3. Find $y = L^{-1}(L(y))$: $y = L^{-1}(\quad) = 2e^{-t} - e^{-2t}$.

Notation for piecewise cont. functions.

$$u_c(t) = \begin{cases} 0 & t<0 \\ 1 & t>c \end{cases} \quad \text{undef. at } t=c.$$

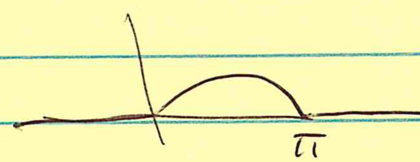
This is called the Heaviside function. There is a heaviside command in most C.A.S.s.

Ex Let $f(t) = \begin{cases} 0 & t<0 \\ 2 & 0<t<2 \\ 1 & t>2. \end{cases}$



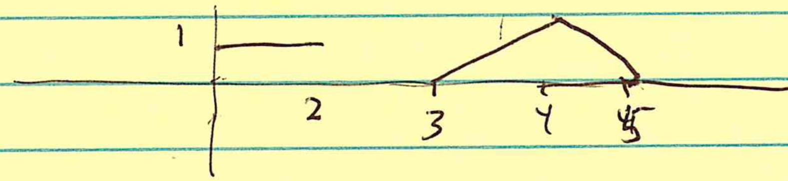
Then $f(t) = 2u_0(t) - u_2(t)$.

Ex Let $g(t) = \begin{cases} 0 & t<0 \\ \sin t & 0<t<\pi \\ 0 & t\geq\pi. \end{cases}$



$$g(t) = \sin t\, u_0(t) - \sin t\, u_\pi(t)$$

Ex $p(t) =$



$$u_0(t) - u_2(t) + (x-3)\left[u_3(t) - u_4(t)\right] +$$

$$(-x+5)\left[u_4(t) - u_5(t)\right]$$

$\boxed{\#12}$ $L(u_c(t)) = \frac{e^{-cs}}{s}$, $s>0$.

Pf: $\int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_c^\infty =$

$\lim_{t\to\infty} \left(-\frac{e^{-st}}{s}\right) - \left(-\frac{e^{-sc}}{s}\right) = e^{-sc}/s$.

"0 for $s>0$

$L(f) = \frac{2}{s} - \frac{e^{-2s}}{s}$

$\boxed{\#13}$ Let $f(t)$ be given and let $F(s) = L(f)$. Then

$$L(u_c(t) f(t-c)) = e^{-sc} F(s).$$

Pf $\int_0^\infty e^{-st} u_c(t) f(t-c)\,dt = \int_c^\infty e^{-st} f(t-c)\,dt =$ Let $r = t-c$. $dr = dt$ $t = r+c$

$= \int_0^\infty e^{-s(r+c)} f(r)\,dr = e^{-sc} \int_0^\infty e^{-sr} f(r)\,dr$

$= e^{-sc} F(s)$

$\boxed{\#14}$ Let $f(t)$ be given and let $F(s) = L(f)$. Then

$$L(e^{ct} f(t)) = F(s-c).$$

Pf: $L(e^{ct} f(t)) = \int_0^\infty e^{-st} e^{ct} f(t)\,dt = \int_0^\infty e^{-(s-c)t} f(t)\,dt$

$= F(s-c)$.

$t \geq 0.$

Ex Solve $y' = f(t)$ $\quad y(0) = 3.$ $\quad f(t)$ was given before.

Sol. Apply L.

$$L(y') = L(f) = \frac{2}{s} - \frac{e^{-2s}}{s}$$

$$sL(y) - \underbrace{y(0)}_{=3} =$$

Solve for $L(y)$ $\quad L(y) = \frac{3}{s} + \frac{2}{s^2} - \frac{e^{-2s}}{s^2}.$

Find $L^{-1}(L(y))$ $\quad L^{-1}\left(\frac{1}{s}\right) = 1 \quad L^{-1}\left(\frac{1}{s^2}\right) = t.$

$L^{-1}\left(e^{-2s} \cdot \frac{1}{s^2}\right)$ uses #13

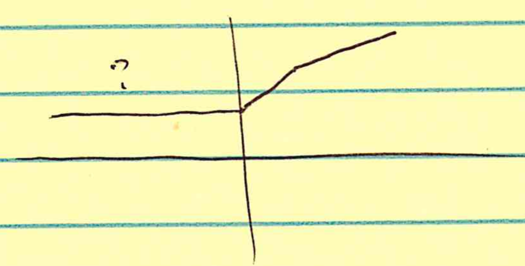
$F(s) = \frac{1}{s^2}$

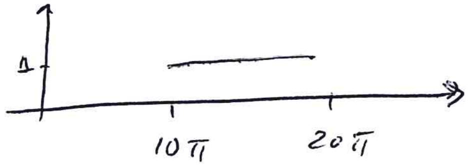
$f(t) = t \qquad c = 2$

$$u_2(t)\, f(t-c) = u_2(t) \cdot (t-2).$$

Thus $y = 3 + 2t - (t-2)\, u_2(t) = \begin{cases} 3+2t & 0 \le t < 2 \\ 5+t & t > 2 \end{cases}$

What about $t < 0$? This does not extend.
But $y' = 0 \Rightarrow y = c$. $y(0) = 3 \Rightarrow y = 3.$



Ex $\quad y'' + y = f(t) =$ 

$y(0) = y'(0) = 0.$

$$f(t) = u_{10\pi}(t) - u_{20\pi}(t)$$

$$L(y'') + L(y) = \frac{e^{-10\pi s} - e^{20\pi s}}{s}$$

$$s^2 L(y) - s y(0) - y'(0) + L(y) = \quad ''$$

$$(s^2+1)\, L(y) = \quad ''$$

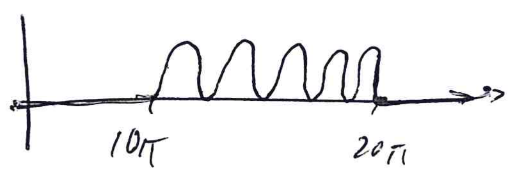
$$L(y) = \frac{e^{-10\pi s} - e^{-20\pi s}}{s(s^2+1)}$$

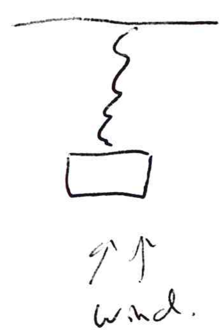
$$L^{-1}\left(e^{-10\pi s} \cdot \frac{1}{s(s^2+1)}\right) = u_{10\pi}(t)\, f(t-10\pi)$$

where $f(t) = L^{-1}\left(\frac{1}{s(s^2+1)}\right) \qquad \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

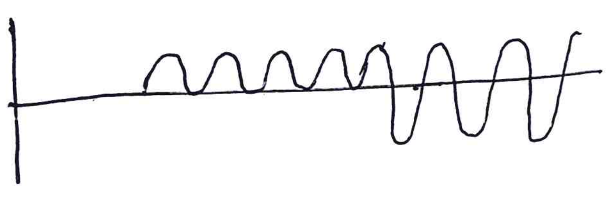
$\Rightarrow f(t) = 1 - \cos(t).$

Thus $\quad y(t) = u_{10\pi}(t)\left(1 - \cos(t - 10\pi)\right) - u_{20\pi}(t)\left(1 - \cos(t - 20\pi)\right).$

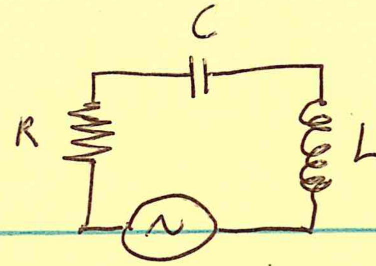




What if we stop at 19π?



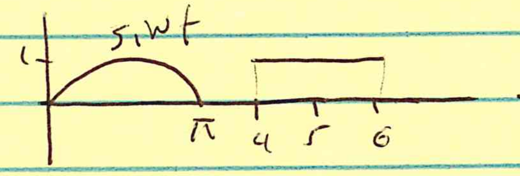
Example



$$L Q'' + R Q' + \frac{1}{C} Q = E(t)$$

where $Q(t)$ is the charge on the capacitor and $E(t)$ is an external applied voltage.

Let $L = 1$, $C = 1$, $R = 10$, $Q'(0) = Q(0) = 0$.

Let $E(t)$ be given by 

Plot $Q(t)$.

Solution $\mathcal{L}\left(LQ'' + RQ' + \frac{1}{C}Q\right) = \mathcal{L}(E(t))$

$$L\,\mathcal{L}(Q'') + R\,\mathcal{L}(Q') + \frac{1}{C}\mathcal{L}(Q) = \mathcal{L}(E(t)).$$

$$L\big(s^2\mathcal{L}(Q) - \underbrace{s\,Q(0) - Q'(0)}_{0}\big) + R\big(s\mathcal{L}(Q) - \underbrace{Q(0)}_{0}\big) + \frac{1}{C}\mathcal{L}(Q)$$

$$= \left(Ls^2 + Rs + \frac{1}{C}\right)\mathcal{L}(Q) = \mathcal{L}(E)$$

Thus

$$Q(t) = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(E)}{Ls^2 + Rs + 1/C}\right).$$

Now we go to the computer!

Laplace Transforms with Maple

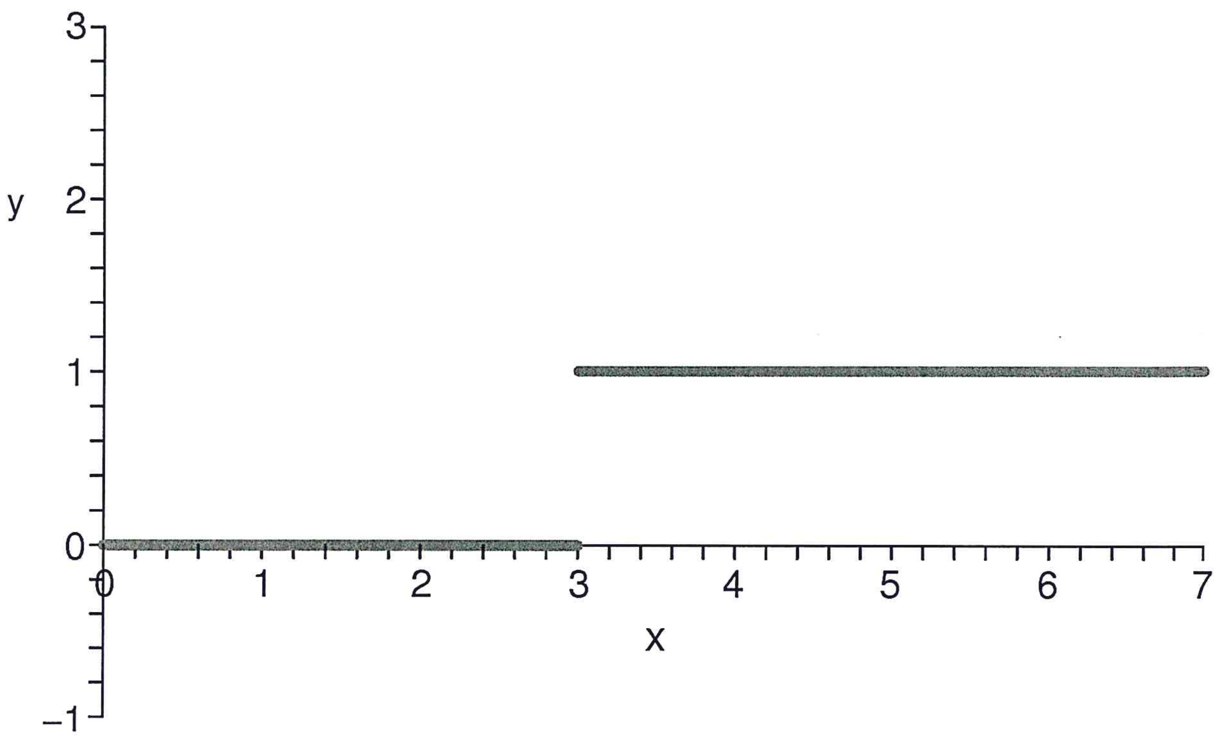
```
> with(inttrans); # Loads a package of commands.
```

[*addtable, fourier, fouriercos, fouriersin, hankel, hilbert, invfourier, invhilbert, invlaplace, invmellin, laplace, mellin, savetable*]

```
> u3:= x-> Heaviside(x-3);
```

$$u3 := x \rightarrow \text{Heaviside}(x-3)$$

```
> plot(u3(x),x=0..7,y=-1..3,discont=true,thickness=3);
```




```
> laplace(u3(t),t,s); # FInd the laplace transform. t is the orgonial
  variable, s is the new new.
```

$$\frac{e^{(-3\,s)}}{s}$$

```
> invlaplace(exp(-3*s)/s,s,t); # FInd the inverse Laplace fransform!
```

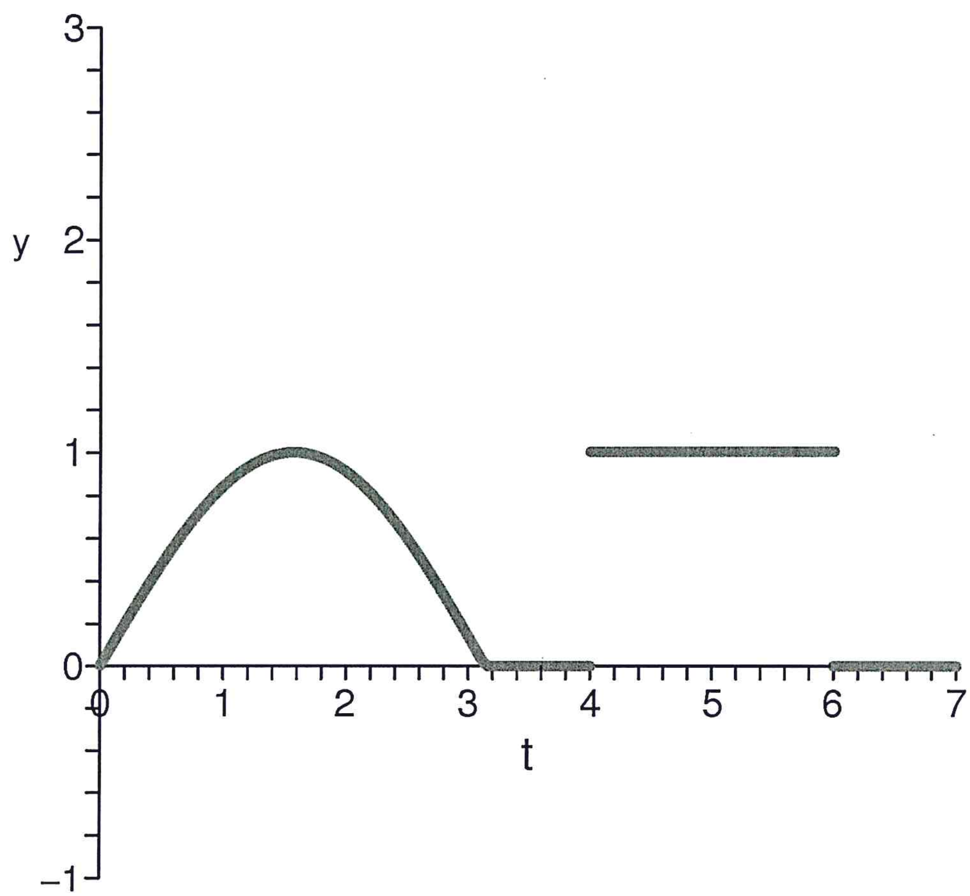
$$\text{Heaviside}(t-3)$$

Next example.

```
> f:= t -> sin(t)*Heaviside(t) - sin(t)*Heaviside(t-Pi)+
  Heaviside(t-4)-Heaviside(t-6);
```

$f := t \to \sin(t)\,\mathrm{Heaviside}(t) - \sin(t)\,\mathrm{Heaviside}(t-\pi) + \mathrm{Heaviside}(t-4) - \mathrm{Heaviside}(t-6)$

```
> plot(f(t),t=0..7,y=-1..3,discont=true,thickness=3);
```




```
> laplace(f(t),t,s);
```

$$\frac{1 + e^{(-s\,\pi)}}{s^2 + 1} + \frac{e^{(-4\,s)} - e^{(-6\,s)}}{s}$$

```
>
```

Now we consider an LRC circuit. Recall the diffeq for the charge on the capacitor is LQ''+RQ'+(1/C)Q = E(t). Will ask what will Q(t) be if we use f(t) above as the external applied voltage. We assume Q'(0)=Q(0)=0. Here are the values for L, R and C.

```
> L:=1;C:=1;R:=10;
```

$$L := 1$$

$$C := 1$$

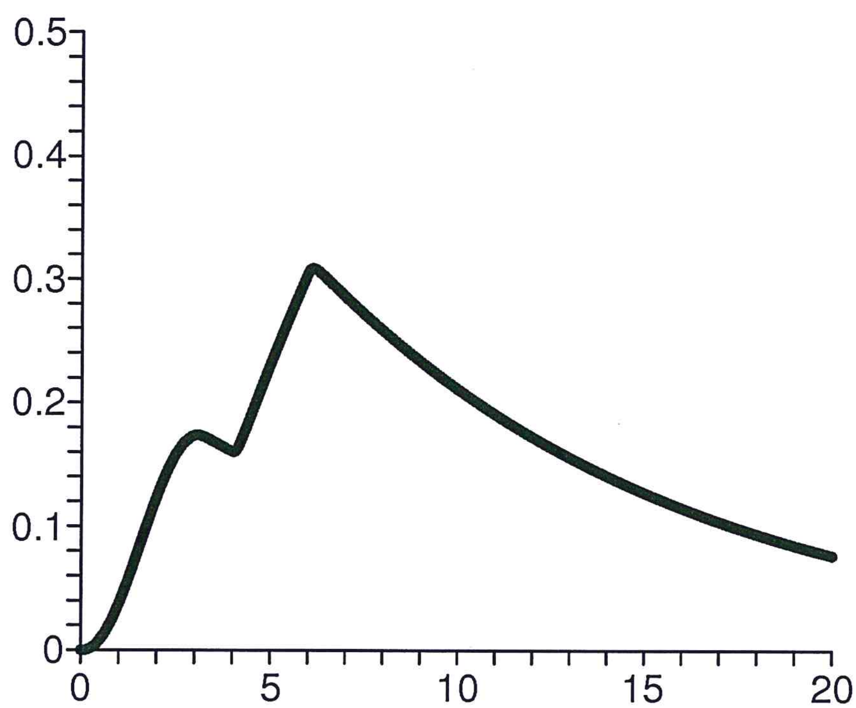
$$R := 10$$

Then we solve Lap(LQ''+RQ'+(1/C)Q + Lap(f(t)) for Lap(Q). We get Lap(Q) = Lap(f)/(Ls^2+Rs+1/C). Then Q(t) = Inverse Lap (Lap Q).

```
> Q:=
  t->invlaplace((((1+exp(-s*Pi))/(s^2+1)+(exp(-4*s)-exp(-6*s))/s)/(L*s^2+R*s+
  1/C),s,t);
```

$$Q := t \to \mathit{invlaplace}\left(\frac{\dfrac{1+e^{(-s\pi)}}{s^2+1}+\dfrac{e^{(-4s)}-e^{(-6s)}}{s}}{L s^2+R s+\dfrac{1}{C}}, s, t\right)$$

```
> plot(Q(t),t=0..20,y=0..0.50,color=blue,thickness=3);
```



```
> evalf(Q(10));
```

$$0.2107079534$$

Let's see what the formula for Q(t) is.

```
> invlaplace(((1+exp(-s*Pi))/(s^2+1)+(exp(-4*s)-exp(-6*s))/s)/(L*s^2+R*s+1)
),s,t);
```

$$-\frac{1}{10}\cos(t)+\frac{1}{24}\left(-24-e^{(-(5+2\sqrt{6})(t-6))}(-12+5\sqrt{6})+e^{((-5+2\sqrt{6})(t-6))}(12+5\sqrt{6})\right)$$

$$\text{Heaviside}(t-6)+\frac{1}{24}\left(24-e^{((-5+2\sqrt{6})(t-4))}(12+5\sqrt{6})+\right.$$

$$e^{\left(\frac{(-2\sqrt{6}-5)\left(-\frac{5t(5+2\sqrt{6})}{-2\sqrt{6}-5}+\frac{20(5+2\sqrt{6})}{-2\sqrt{6}-5}-\frac{2t\sqrt{6}(5+2\sqrt{6})}{-2\sqrt{6}-5}+\frac{8\sqrt{6}(5+2\sqrt{6})}{-2\sqrt{6}-5}\right)}{5+2\sqrt{6}}\right)}\Big($$

$$\left.-12+5\sqrt{6}\Big)\right)\text{Heaviside}(t-4)+\frac{1}{120}\text{Heaviside}(t-\pi)\,(12\cos(t)$$

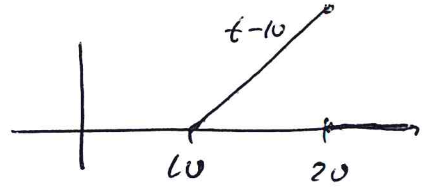
$$+e^{(-5t+5\pi)}(5\sqrt{6}\sinh(2(t-\pi)\sqrt{6})+12\cosh(2(t-\pi)\sqrt{6})))$$

$$+\frac{1}{120}e^{(-5t)}(12\cosh(2t\sqrt{6})+5\sqrt{6}\sinh(2t\sqrt{6}))$$

Yuck!

```
>
```

Ex $y'' + 3y' + 2y = f(t) =$



$y(0) = y'(0) = 0.$

$$f(t) = u_{10}(t-10) - u_{20}(t)(t-10)$$

Note: homogeneous solution is $y_h = c_1 e^{-2t} + c_2 e^{-t} \to 0$ as $t \to \infty$.
So, what do we expect solution of our problem to look like?

$$L(y'' + 3y' + 2y) = s^2 L(y) + 3s L(y) + 2L(y) = (s^2+3s+2)L(y).$$

$$L(u_{10}(t)(t-10)) = e^{-10s}F(s), \quad \text{where } F(s) = L(t) = \frac{1}{s^2}$$

$$L(u_{20}(t)(t-10)) = e^{-20s}F(s), \quad \text{where } F(s) = L(t+10) = \frac{1}{s^2} + \frac{10}{s}$$

Thus $L(f) = e^{-10s}\left[\frac{1}{s^2}\right] - e^{-20s}\left[\frac{1}{s^2} + \frac{10}{s}\right].$

And,

$$L(y) = \frac{e^{-10s}\left[\frac{1}{s^2}\right] - e^{-20s}\left[\frac{1}{s^2} + \frac{10}{s}\right]}{(s^2+3s+2)}$$

To find y I used a computer to ~~compute~~ find ~~and graph~~ the inverse Laplace transform and then to graph $y(t)$.

See next page.

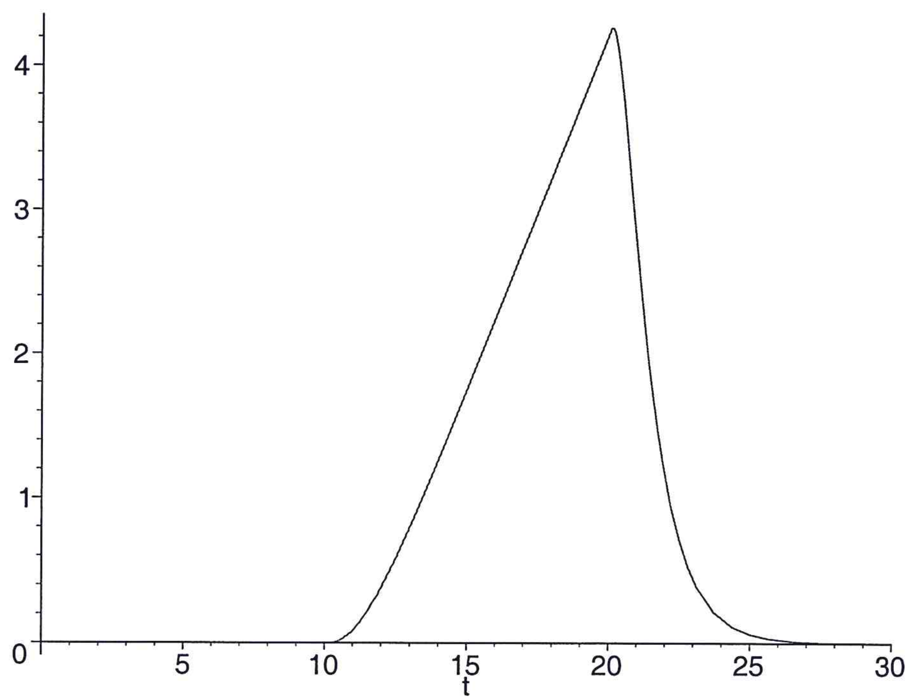
```
> with(inttrans):
> invlaplace((exp(-10*s)*(1/s^2) - exp(-20*s)*(1/s^2 +
  10/s))/(s^2+3*s+2),s,t);
```

$$\frac{1}{2}\text{Heaviside}(t-10)\,t-\frac{23}{4}\text{Heaviside}(t-10)-\frac{1}{4}\text{Heaviside}(t-10)\,\mathbf{e}^{(-2\,t+20)}$$
$$+\text{Heaviside}(t-10)\,\mathbf{e}^{(-t+10)}-\frac{1}{2}\text{Heaviside}(t-20)\,t+\frac{23}{4}\text{Heaviside}(t-20)$$
$$-\frac{19}{4}\text{Heaviside}(t-20)\,\mathbf{e}^{(-2\,t+40)}+9\,\text{Heaviside}(t-20)\,\mathbf{e}^{(-t+20)}$$

```
> f:= t -> (1/2*t-23/4-1/4*exp(-2*t+20)+exp(-t+10))*Heaviside(t-10)
  +
  (-1/2*t+23/4-19/4*exp(-2*t+40)+9*exp(-t+20))*Heaviside(t-20);
>
```

$$f:=t\rightarrow\left(\frac{1}{2}t-\frac{23}{4}-\frac{1}{4}\mathbf{e}^{(-2\,t+20)}+\mathbf{e}^{(-t+10)}\right)\text{Heaviside}(t-10)$$
$$+\left(-\frac{1}{2}t+\frac{23}{4}-\frac{19}{4}\mathbf{e}^{(-2\,t+40)}+9\,\mathbf{e}^{(-t+20)}\right)\text{Heaviside}(t-20)$$

```
> plot(f(t),t=0..30);
```




```
>
```

6.3 #28

Let $f(t)$ be periodic with period T. (That is $f(t+T) = f(t)$.) Show that

$$L(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Solution

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt =$$

$$= \sum_{n=0}^{\infty} \int_0^T e^{-s(t+nT)} f(t+nT)\,dt =$$

$$= \left(\sum_{n=0}^{\infty} e^{-snT}\right) \int_0^T e^{-st} f(t)\,dt$$

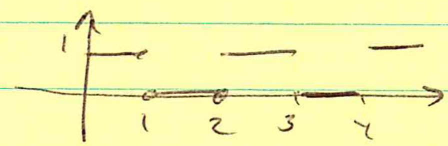
$$\sum_{n=0}^{\infty} \left(e^{-sT}\right)^n = \frac{1}{1-e^{-sT}}$$

, since it is a geometric series.

But this gives the desired result,

$$L(f) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

Ex Solve $y'' + y = g(t) \quad y'(0) = 0 \quad y(0) = 0,$

where $g(t) =$ 

, a square wave.

plot $y(t)$ for $t \in (0, 30)$.

Solution Apply Laplace Transform.

$$s^2 L(y) + L(y) = L(g).$$

$$L(y) = \frac{L(g)}{s^2+1}$$

To find $L(g)$ use # 28 in 6.3

"$\int_0^2 e^{-st} f(t)\,dt$ "

$$L(g) = \frac{\int_0^1 e^{-st}\,dt}{1 - e^{-2s}} = \frac{-\frac{1}{s} e^{-s} + \frac{1}{s}}{1 - e^{-2s}} = \frac{1}{s} \frac{1 - e^{-s}}{1 - e^{-2s}}$$

Simplify: $L(g) = \frac{1}{s} \frac{1}{1 + e^{-s}}$.

So, $L(y) = \frac{1}{s(s^2+1)} \cdot \frac{1}{1+e^{-s}}$. $\left(\text{Let } F(s) = \frac{1}{s(s^2+1)}\right)$

To find inverse use $\frac{1}{1+e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + e^{-4s} - \cdots$
and Table 6.2.1 entry 13:

$$L^{-1}\left(e^{-ns} F(s)\right) = u_n(t) f(t-n).$$

For us $F(s) = \frac{1}{s(s^2+1)}$.

Let $f(t) = L^{-1}(F(s))$. Compute

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$ Thus

$f(t) = 1 - \cos t$, from Table 6.2.1.

Thus,

$$y(t) = \sum_{n=0}^{\infty} L^{-1}\left(e^{-ns} F(s)\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n u_n(t)\left(1 - \cos(t-n)\right)$$

```
> f:= t -> 1-cos(t);
```

$$f := t \to 1 - \cos(t)$$

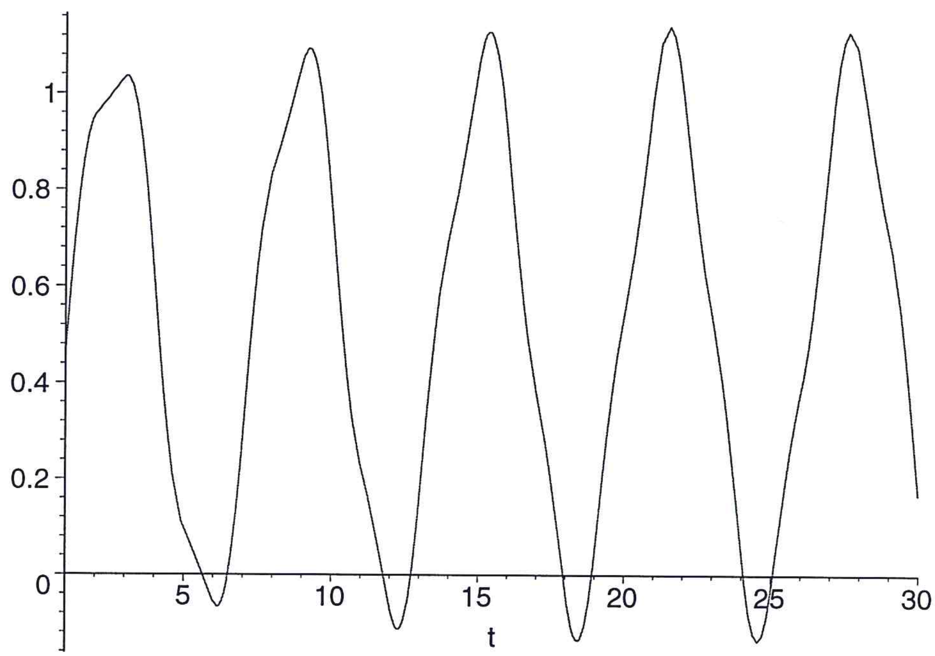
```
> u:= (t,n) -> Heaviside(t-n);
```

$$u := (t, n) \to \mathrm{Heaviside}(t - n)$$

```
> y := (t,k) -> sum((-1)^n*u(t,n)*f(t-n),n=0..k);
```

$$y := (t, k) \to \sum_{n=0}^{k} (-1)^n \, \mathrm{u}(t, n) \, \mathrm{f}(t - n)$$

```
> plot(y(t,31),t=1..30);
```

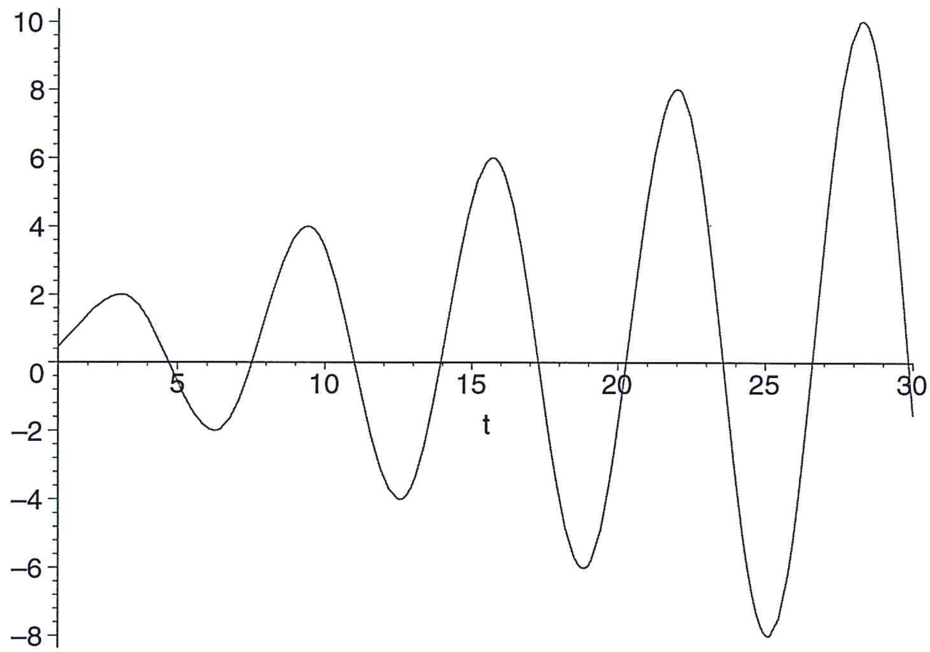


Redo with period of sq wave $= 2\pi$.

```
> y := (t,k) -> sum((-1)^n*u(t,Pi*n)*f(t-Pi*n),n=0..k);
```

$$y := (t, k) \to \sum_{n=0}^{k} (-1)^n \, \mathrm{u}(t, \pi n) \, \mathrm{f}(t - \pi n)$$

```
> plot(y(t,31),t=1..30);
```


10
8
6
4
2
0
–2
–4
–6
–8
5
10
15
20
25
30
t

[>