1. [20 points] Find the general solution to each of the following.
   a. $2y' = e^{t/3} - y$.
   b. $(y \cos x) + (\sin x)y' = 0$.
   c. $y'' + 2y' - 15y = 0$.
   d. $y'' + 2y' + 2y = 0$.

2. [20 points] Solve the initial value problem
   $$t^2y'' + 2ty' = 1 \quad (t > 0), \quad y(1) = 3, \quad y'(1) = 2.$$

3. [20 points] Let $F(y) = \begin{cases} 
   y \ln \left(\frac{y^3}{27}\right) (y^3 - 27) & y \neq 0 \\
   0 & y = 0.
\end{cases}$
   Now consider the autonomous differential equation
   $$\frac{dy}{dt} = F(y).$$
   Find the equilibrium solutions and determine their stability types. Draw the phase portrait sketch several solution curves. Indicate (roughly) their concavity. Hint: Remember $\ln 1 = 0$. 

\[\begin{array}{c|c|c}
y(t) & 4 & 3 \\
& 2 & 1 \\
& 0 & t \\
& -1 & -2 \\
& -3 & -4
\end{array}\]
Bon: Show that \( F(y) \) is in fact continuous at \( y = 0 \). Do this by studying the limit of \( F(y) \) as \( y \to 0 \). You’ll need L’Hopital’s Rule.

4. [20 points] Consider the equation \( y' = 1 + t^2 - 2ty + y^2 \). This is an example of a Riccati equation.

a. Show that \( y(t) = t \) is a solution.

b. Let \( y = t + \frac{1}{v(t)} \). Substitute this into the given Riccati equation and derive a new differential equation in \( v \). Solve this equation for \( v(t) \). (There will be one arbitrary constant in your solution for \( v(t) \).)

Thus you have found a family of additional solutions to the original Riccati equation!

5. [20 points] Let

\[
 f(t) = \begin{cases} 
 t & t \in [0, 5\pi] \\
 2 & t > 5\pi.
\end{cases}
\]

Find a differentiable solution to the initial value problem

\[
y'' + y = f(t) \quad (t \geq 0),
\]

\( y(0) = 0 \) and \( y'(0) = 1 \).

6. [20 points] Find the general power series solution to

\[
 xy'' + (x + 2)y' + 3y = 0
\]

centered about \( x = 0 \). Find a recursive formula for \( a_n \).

Actually this problem is ill posed. The radius of convergence is 0.

7. [20 points] Consider \( y'' + e^x y' - (3 \sin x) y = 0 \). Suppose \( y(0) = 1 \) and \( y'(0) = 2 \). Find the first five terms of the power series solution. What can you say about its radius of convergence?

8. [20 points] a. Graph the square wave given by

\[
 f(x) = \begin{cases} 
 1 & x \in [n, n+1) \text{ for } n \text{ even} \\
 -1 & x \in [n, n+1) \text{ for } n \text{ odd},
\end{cases}
\]

over \( x \in [-3, 3] \).

b. Find the Fourier Series of \( f(x) \).

9. [20 points] The heat equation does not account for the fact that heat may dissipate from the material to the surrounding environment. Let \( A \) the ambient temperature that a metal rod is in and assume it is constant.

We know from Newton’s Law of Cooling that heat flow is proportional to the difference between \( u(x, t) \) and \( A \). This gives the PDE
\[ u_t = au_{xx} - \gamma(u - A). \]

Convince yourself that this equation is not separable. But make the substitution \( w(x, t) = u(x, t) - A \) and derive a PDE in \( w \). Show that this equation is separable.

10. [20 points] Wave equation can be applied to traveling waves. Here the string is infinitely long. So, there are no boundary conditions. We will let \( f(x) \) be some initial configuration and assume the initial speed of motion is zero. Thus we have,

\[ a^2 u_{xx} = u_{tt}, \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0. \]

a. Let \( \phi \) be any differentiable function. Show that \( u(x, t) = \phi(x \pm at) \) satisfy the wave equation; that is show \( a^2 \partial_{xx} \phi(x \pm at) = \partial_{tt} \phi(x \pm at) \).

b. Let \( u(x, t) = \frac{1}{2}(f(x - at) + f(x + at)) \). Show that this satisfies the wave equation and the two initial conditions.

c. Let \( f(x) \) be given by the graph below. Let \( a = 1 \). Graph \( u(x, 2) \) where \( u \) is the solution given in part b. Then graph \( u(x, 3) \) and \( u(x, 4) \). It is a traveling wave!

![Graph of initial configuration](attachment:image.jpg)