Name: ______________________ Section time: ______________________

SCIENTIFIC CALCULATORS ALLOWED

1. [20 points] Find the general solution to each of the following.
   a. $2y' = e^{t/3} - y$.
   b. $(y\cos x) + (\sin x)y' = 0$.
   c. $y'' + 2y' - 15y = 0$.
   d. $y'' + 2y' + 2y = 0$.

2. [20 points] Solve the initial value problem

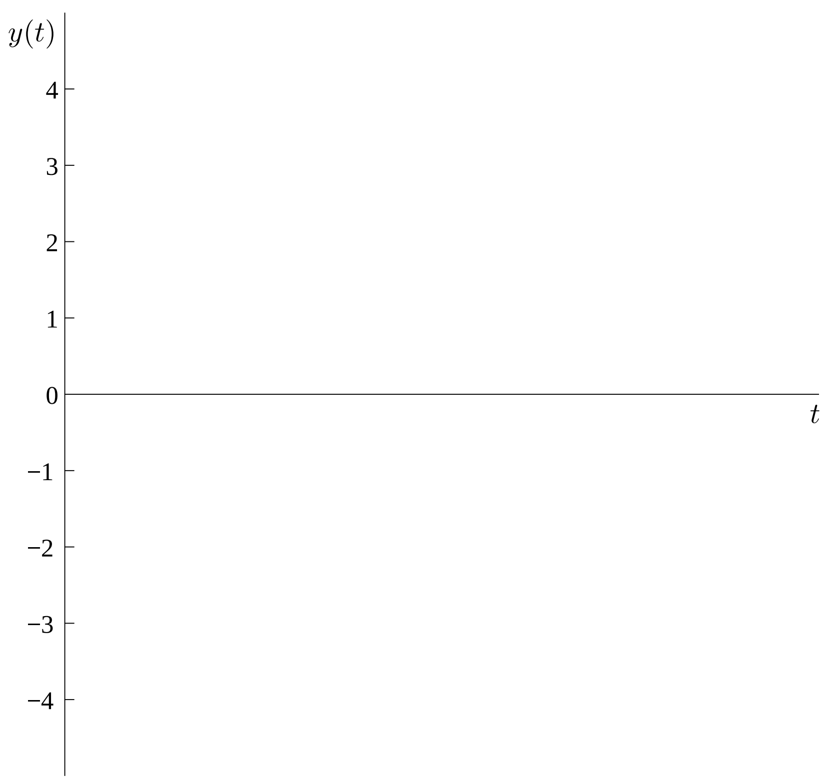
$$t^2y'' + 2ty' = 1 \ (t > 0), \qquad y(1) = 3,\ y'(1) = 2.$$

3. [20 points] Let $F(y) = \begin{cases} y\ln\left(\frac{|y|}{2}\right)(y^3 - 27) & y \neq 0 \\ 0 & y = 0. \end{cases}$

   Now consider the autonomous differential equation

$$\frac{dy}{dt} = F(y).$$

   Find the equilibrium solutions and determine their stability types. Draw the phase portrait sketch several solution curves. Indicate (roughly) their concavity. Hint: Remember $\ln 1 = 0$.



Bonus: Show that $F(y)$ is in fact continuous at $y = 0$. Do this by studying the limit of $F(y)$ as $y \to 0$. You'll need L'Hopital's Rule.

4. [20 points] Consider the equation $y' = 1 + t^2 - 2ty + y^2$. This is an example of a *Riccatti equation.*

   a. Show that $y(t) = t$ is a solution.

   b. Let $y = t + \frac{1}{v(t)}$. Substitute this into the given Riccatti equation and derive a new differential equation in v. Solve this equation for $v(t)$. (There will be one arbitrary constant in your solution for $v(t)$.) Thus you have found a family of additional solutions to the original Riccatti equation!

5. [20 points] Let

$$f(t) = \begin{cases} t & t \in [0, 5\pi] \\ 2 & t > 5\pi. \end{cases}$$

   Find a differentiable solution to the initial value problem

$$y'' + y = f(t) \quad (t \geq 0),$$

   $y(0) = 0$ and $y'(0) = 1$.

6. [20 points] Find the general power series solution to

$$xy'' + (x + 2)y' + 3y = 0$$

   centered about $x = 0$. Find a recursive formula for a_n.

   Actually this problem is ill posed. The radius of convergence is 0.

7. [20 points] Consider $y'' + e^x y' - (3 \sin x)y = 0$. Suppose $y(0) = 1$ and $y'(0) = 2$. Find the first five terms of the power series solution. What can you say about its radius of convergence?

8. [20 points] a. Graph the square wave given by

$$f(x) = \begin{cases} 1 & x \in [n, n+1) \text{ for } n \text{ even} \\ -1 & x \in [n, n+1) \text{ for } n \text{ odd}, \end{cases}$$

   over $x \in [-3, 3]$.

   b. Find the Fourier Series of $f(x)$.

9. [20 points] The heat equation does not account for the fact that heat may dissipate from the material to the surrounding environment. Let A the ambient temperature that a metal rod is in and assume it is constant.

   We know from Newton's Law of Cooling that heat flow is proportional to the difference between $u(x, t)$ and A. This gives the PDE

$$u_t = au_{xx} - \gamma(u - A).$$

Convince yourself that this equation is not separable. But make the substitution $w(x,t) = u(x,t) - A$ and derive a PDE in w. Show that this equation is separable.

10. [20 points] Wave equation can be applied to traveling waves. Here the string is infinitely long. So, there are no boundary conditions. We will let $f(x)$ be some initial configuration and assume the initial speed of motion is zero. Thus we have,

$$a^2 u_{xx} = u_{tt}, \quad u(x,0) = f(x), \quad u_t(x,0) = 0.$$

a. Let ϕ be any differentiable function. Show that $u(x,t) = \phi(x \pm at)$ satisfy the wave equation; that is show $a^2 \partial_{xx}\phi(x \pm at) = \partial_{tt}\phi(x \pm at)$.

b. Let $u(x,t) = \frac{1}{2}(f(x - at) + f(x + at))$. Show that this satisfies the wave equation and the two initial conditions.

c. Let $f(x)$ be given by the graph below. Let $a = 1$. Graph $u(x,2)$ where u is the solution given in part b. Then graph $u(x,3)$ and $u(x,4)$. It is a traveling wave!