Applications (2.3)

I. Mixing problems.

Ex 1 A tank has 100 gal of water. Initially, the water is fresh, that is there is no salt in it. At $t=0$ we start pouring in salt water with 1 lb/gal of salt at a rate of 2 gal/min. Simultaneously, water pours out the same rate. Assume the water in the tank is kept well mixed. How many pounds of salt are in the tank when $t = 30$ min?

Solution Step up a model. Let $Q(t)$ be the quatity of salt in the tank at time t in minutes, in pounds. Then

$$\frac{dQ}{dt} = \left\{\begin{matrix}\text{rate}\\ \text{in}\end{matrix}\right\} - \left\{\begin{matrix}\text{rate}\\ \text{out}\end{matrix}\right\}$$

$$= 1\,\frac{\text{lb}}{\text{gal}} \cdot 2\,\frac{\text{gal}}{\text{min}} - \frac{Q(t)\,\text{lb}}{100\,\text{gal}} \cdot 2\,\frac{\text{gal}}{\text{min}}$$

$$= 2 - \frac{Q}{50} \quad (\text{lbs}/\text{min}).$$

This together with $Q(0)=0$ gives an initial value problem.

Step 2 Solve the initial value problem

$$Q' = 2 - \frac{Q}{50}, \quad Q(0) = 0.$$

It is linear and separable. I'll use the latter.

$$\frac{-50}{-50}\int \frac{dQ}{2-\frac{Q}{50}} = \int dt$$

$$-50 \ln\left|2 - \frac{Q}{50}\right| = t + C$$

$$\ln\left|2 - \frac{Q}{50}\right| = -\frac{1}{50}t + C$$

$$\left|2 - \frac{Q}{50}\right| = C e^{-\frac{1}{50}t} \qquad C > 0$$

$$2 - \frac{Q}{50} = C e^{-\frac{1}{50}t}$$

$$Q(t) = 100 - C e^{-\frac{1}{50}t}$$

Since $Q(0) = 0$, we get that $C = 100$.

Thus $\quad Q(t) = 100 - 100 e^{-\frac{t}{50}}$.

Step 3 Plug in $t = 30$.

$$Q(30) = 100\left(1 - e^{-\frac{30}{50}}\right) \approx \boxed{45.1188 \text{ lbs.}}$$

Ex 2 A tank originally contains 200 gal of fresh-water. Then water containing 0.5 lb/gal of salt is pumped in at 3 gal/min, while the well mixed solution leaves at 2 gal/min. How many pounds of salt are in the tank after 10 min?

Solution Step 1, set up model. Let $Q(t)$ be the quantity of salt in the tank in pounds.

$$\frac{dQ}{dt} = \begin{Bmatrix}\text{rate}\\ \text{in}\end{Bmatrix} - \begin{Bmatrix}\text{rate}\\ \text{out}\end{Bmatrix}$$

$$= 0.5\,\frac{\text{lb}}{\text{gal}} \times 3\,\frac{\text{gal}}{\text{min}} - \frac{Q(t)\ \text{lb}}{200+t\ \text{gal}}\,2\,\frac{\text{gal}}{\text{min}}$$

$$= 1.5 - \frac{2Q}{200+t}.$$

$Q(0)=0$ is given.

Step 2 Solve the diff eqn with init. cond.

$$Q' - \frac{2Q}{200+t} = 1.5 \quad \text{is linear.}$$

$$\mu = e^{\int \frac{2}{200+t}\,dt} = e^{2\ln|200+t|} = (200+t)^2.$$

Thus, $(200+t)^2 Q' + 2(200+t)\,Q = 1.5(200+t)^2.$

or, $$\left[(200+t)^2 Q\right]' = 1.5\,(200+t)^2.$$

Integration gives

$$(200+t)^2 Q = 1.5 \int 40{,}000 + 400t + t^2 \, dt$$

$$= 1.5\left(40{,}000t + 200t^2 + \frac{t^3}{3}\right) + C$$

Thus $$Q(t) = \frac{1.5t\left(40{,}000 + 200t + \frac{1}{3}t^2\right) + C}{(200+t)^2}.$$

Since $Q(0)=0$ we have $\frac{C}{(200)^2} = 0 \Rightarrow C=0.$

Thus,
$$Q(t) = \frac{1.5t\left(40{,}000 + 200t + \frac{1}{3}t^2\right)}{(200+t)^2}.$$

$\boxed{\text{Step 3}}$ Plug in!

$$Q(10) = \frac{15\left(40{,}000 + 2{,}000 + 33\frac{1}{3}\right)}{(210)^2} = \frac{630{,}500}{44{,}100}$$

$$\boxed{\approx 14.29705 \text{ lb}}$$

Ex 3 A tank initially contains 100 gals of freshwater. From time $t=0$ to $t=10$ min saltwater with 0.2 lb/gal salt flows into the tank at 2 gal/min, while the well mixed solution drains out at the same rate. From $t=10$ to $t=20$ min saltwater with 0.3 lb/gal salt flows into the tank at 1 gal/min, while the well mixed solution drains out at the same rate. How many pounds of salt are in the tank after 20 minutes?

Solution We need two models, one for $0 \le t \le 10$ and another for $10 \le t \le 20$. Again let $Q(t)$ be the pounds of salt in the tank at time t, measured in minutes.

First 10 minutes

$$Q' = \left\{ \begin{matrix} \text{rate} \\ \text{in} \end{matrix} \right\} - \left\{ \begin{matrix} \text{rate} \\ \text{out} \end{matrix} \right\} = 2(0.2) - \frac{Q(t)}{100} \cdot 2$$

$$Q' + \frac{1}{50} Q = 0.4$$

have students do this

$$Q = 20\left(1 + Ce^{-t/50}\right).$$

Then $Q(0)=0$ implies $C=-1$.

Now $\quad Q(t) = 20\left(1 - e^{-t/50}\right).$

Thus $\quad Q(10) = 20\left(1 - e^{-1/5}\right) \approx 3.625384938$

<u>Second 10 minutes</u>

$$Q' = 0.3 \cdot 1 - \frac{Q(t)}{100} \cdot 1 \quad \leftarrow \text{one}$$

or $\quad Q' + \frac{1}{100} Q = 0.3$

students do this.

Thus $\quad Q(t) = 30\left(1 - Ce^{-t/100}\right)$

We know $\quad Q(10) = 20\left(1 - e^{-\frac{1}{5}}\right)$. Use this to get C.

$$30\left(1 - Ce^{-1/10}\right) = 20\left(1 - e^{-\frac{1}{5}}\right)$$

$$C = \left[1 - \tfrac{2}{3}\left(1 - e^{-1/5}\right)\right] e^{\frac{1}{10}}$$

Then $\quad Q(20) = 30\left(1 - Ce^{-\frac{1}{5}}\right) \approx \boxed{6.135261406 \text{ lb}}$

The next example on radioactive decay is an example of <u>coupled systems</u> of equations

<u>Ex 4</u> Substance A decays into substance B with a half-life h_1. B decays into C with a half-life h_2. Assume C is stable. Let A_0, B_0 and C_0 be the initial amounts of A, B and C, resp. Find formulas for $A(t)$, $B(t)$ and $C(t)$ and find the maximum amount of B and the time this occurs.

<u>Sol</u> We apply the idea $\frac{dQ}{dt} = \left\{ \begin{matrix} \text{rate} \\ \text{in} \end{matrix} \right\} - \left\{ \begin{matrix} \text{rate} \\ \text{out} \end{matrix} \right\}$ to A, B and C.

$$\frac{dA}{dt} = 0 - r_1 A \qquad \text{where } r_1 = \frac{\ln 2}{h_1}$$

$$\frac{dB}{dt} = r_1 A - r_2 B \qquad \text{where } r_2 = \frac{\ln 2}{h_2}.$$

$$\frac{dC}{dt} = r_2 B - 0.$$

<u>A</u> $A = A_0 e^{-r_1 t}$ as we already know.

<u>B</u> We have $B' + r_2 B = r_1 A_0 e^{-r_1 t}$. Let $\mu = e^{r_2 t}$.

$$\text{Then} \qquad (B e^{r_2 t})' = r_1 A_0 e^{(r_2 - r_1)t}.$$

$$B e^{r_2 t} = \frac{r_1}{r_2 - r_1} A_0 e^{(r_2 - r_1)t} + D$$

Don't want to use C.

Thus, $B(t) = \frac{r_1}{r_2 - r_1} A_0 e^{-r_1 t} + D e^{-r_2 t}$

Now we find the constant D, for $B(0) = B_0$.

$$B(0) = \frac{r_1}{r_2 - r_1} A_0 + D = B_0 .$$

Thus $D = B_0 - \frac{r_1}{r_2 - r_1} A_0$.

Hence $B(t) = \frac{r_1}{r_2 - r_1} A_0 e^{-r_1 t} + \left(B_0 - \frac{r_1}{r_2 - r_1} A_0 \right) e^{-r_2 t}$.

<u>max B</u> To find max of $B(t)$ compute $B'(t)$ and set $= 0$.

$$B'(t) = \frac{-r_1^2}{r_2 - r_1} A_0 e^{-r_1 t} - r_2 \left(B_0 - \frac{r_1 A_0}{r_2 - r_1} \right) e^{-r_2 t} = 0$$

Mult. through by $e^{r_2 t}$.

$$\frac{-r_1^2}{r_2 - r_1} A_0 e^{(r_2 - r_1)t} = r_2 \left(B_0 - \frac{r_1 A_0}{r_2 - r_1} \right)$$

$$e^{(r_2 - r_1)t} = \frac{r_2 \left(B_0 - \frac{r_1 A_0}{r_2 - r_1} \right)}{\frac{-r_1^2}{r_2 - r_1} A_0}$$

Thus $t_{max} = \frac{1}{r_2 - r_1} \ln \left(\frac{r_2 \left(B_0 - \frac{r_1 A_0}{r_2 - r_1} \right)}{\frac{-r_1^2}{r_2 - r_1} A_0} \right)$

t at max

If $t_{max} > 0$, there will be a max at $t = t_{max}$.
If $t_{max} \le 0$, then the max of $B(t)$ is at $t = 0$ and hence is B_0.

$B_{max} = B(t_{max})$.

Let's do a simple case with numbers. Let

$$A_0 = 10, \quad B_0 = 5, \quad r_1 = 1, \quad r_2 = 2.$$

Then

$$t_{max} = 1 \cdot \ln\left(\frac{2(5-10)}{-10}\right) = \ln(1) = 0.$$

The graph is on the next page.

Now let's try $A_0 = 10$, $B_0 = 0$, $r_1 = 2$, $r_2 = 1$.

Then

$$t_{max} = -\ln\left(\frac{0+20}{4 \cdot 10}\right) = -\ln\left(\tfrac{1}{2}\right) = \ln 2 = 0.6931$$

$$B(t) = -20\,e^{-2t} + 20\,e^{-t} = 20\left(e^{-t} - e^{-2t}\right).$$

$$B(\ln 2) = 20\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = 5.$$

The graph is on the next page.

C <u>Don't Forget $C(t)$</u>!

$$C' = r_2 B \Rightarrow C(t) = \int r_2 B(t)\, dt =$$

$$\int \frac{r_1 r_2}{r_2 - r_1} A_0 e^{-r_1 t} + r_2\left(B_0 - \frac{r_1}{r_2 - r_1} A_0\right) e^{-r_2 t}\, dt$$

$$= \frac{-r_2}{r_2 - r_1} A_0 e^{-r_1 t} - \left(B_0 - \frac{r_1}{r_2 - r_1} A_0\right) e^{-r_2 t} + E.$$

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> plot(10*exp(-t)-5*exp(-2*t),t=-1..2, color=black, thickness=4);
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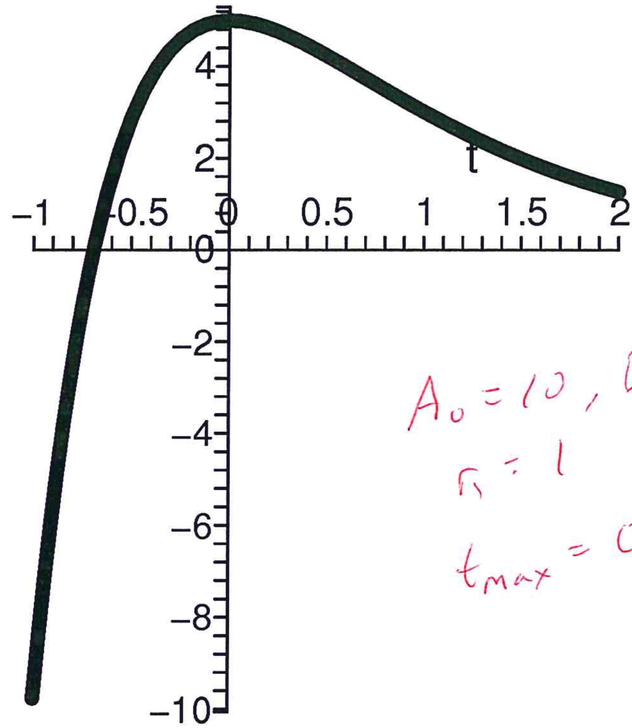

$A_0 = 10,\ B_0 = 5$

$r_1 = 1 \quad r_2 = 2$

$t_{max} = 0 \quad B_{max} = B_0 = 5.$

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> plot(20*(exp(-t)-exp(-2*t)),t=0..2, color=black, thickness=4);
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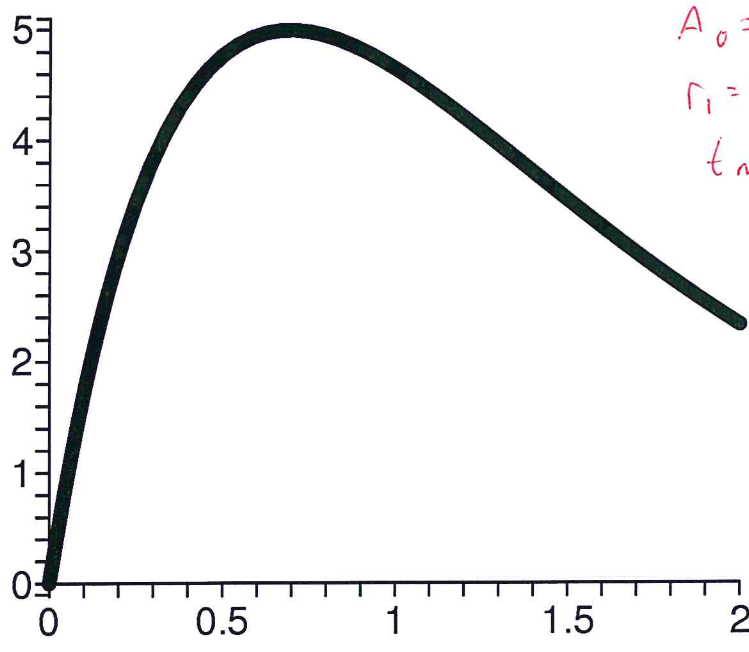


$A_0 = 10,\ B_0 = 0$

$r_1 = 2 \quad r_2 = 1$

$t_{max} = \ln 2,\ B_{max} = 5$

4P

Since $C(0) = C_0$ we get

$$\frac{-r_2}{r_2 - r_1} A_0 - \left(B_0 - \frac{r_1}{r_2 - r_1} A_0\right) + E = C_0$$

Thus $E = C_0 + \frac{r_2}{r_2 - r_1} A_0 - \left(B_0 - \frac{r_1}{r_2 - r_1} A_0\right)$

$$= C_0 + B_0 + \frac{r_2 - r_1}{r_2 - r_1} A_0 = C_0 + B_0 + A_0$$

Thus

$$C(t) = A_0 + B_0 + C_0 - \frac{r_2}{r_2 - r_1} A_0 e^{-r_1 t} - \left(B_0 - \frac{r_1}{r_2 - r_1} A_0\right) e^{-r_2 t}.$$

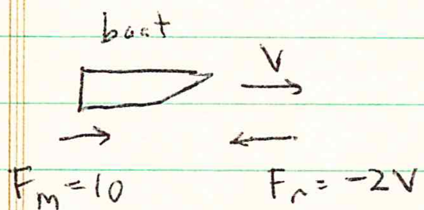
Notice: $\lim_{t \to \infty} C(t) = A_0 + B_0 + C_0.$

Motion Problems with Resistance

Recall $F = ma = m\frac{dv}{dt}$.

Ex 5 A boat weighing 480 lb is pushed forward by a 10 lb force (from its motor). It starts from rest. The water resistance force is twice the speed of the boat. What speed will boat tend toward?

Sol.



$$F = \underset{\text{motor}}{F_m} + \underset{\text{resistance}}{F_r} = 10 - 2V.$$

Thus, $$\frac{dV}{dt} = \frac{10-2V}{m}.$$

But mass $\neq$ weight. $m = \frac{W}{g} = \frac{480}{32} = 15$ slugs.

Now we have $V' + \frac{2}{15}V = \frac{2}{3}$. Also $V(0) = 0$.

Find the solution. Let $\mu = e^{\frac{2}{15}t}$. Multiply through.

$$e^{\frac{2}{15}t}V' + \frac{2}{15}e^{\frac{2}{15}t}V = \frac{2}{3}e^{\frac{2}{15}t}$$

$$\left(e^{\frac{2}{15}t}V\right)' = \frac{2}{3}e^{\frac{2}{15}t}$$

$$e^{\frac{2}{15}}V = \frac{2}{3}\frac{15}{2}e^{\frac{2}{15}t} + C$$

$$v = 5 + Ce^{-\frac{2}{15}t}.$$

At $t=0$, $v=0$ so $C=-5$.

Thus $v(t) = 5\left(1 - e^{-\frac{2}{15}t}\right)$.

Now we can see how to answer the question.

$$\lim_{t \to \infty} v(t) = 5.$$

You should check that the units are ft/sec.

Ex 6 An object with initial velocity $v(0)=v_0$ in the downward direction is subject to the forces of gravity and air resistance. The latter is proportional to $\sqrt{v}$. Find the terminal velocity. (Assume $v(t) \geq 0$, where positive means downward.

Sol. $F = ma \Rightarrow mv' = mg - k\sqrt{v}$, where $k>0$ is the proportionality constant. Thus our model is

$$\frac{dv}{dt} = g - \frac{k}{m}\sqrt{v}, \quad v(0)=v_0.$$

Thus
$$\int \frac{dv}{g - \frac{k}{m}\sqrt{v}} = \int dt$$

Rewrite as
$$(*) \; \int \frac{dv}{\frac{mg}{k} - \sqrt{v}} = \int \frac{k}{m}\, dt = \frac{k}{m}t + C = (\#).$$

Let $a = \frac{mg}{k}$. Let $u = a - \sqrt{v}$. Then $du = \frac{-1}{2\sqrt{v}}\, dv$.

So, $dv = -2\sqrt{v}\, du = 2(u-a)\, du$. Thus,

$$(*) = 2\int \frac{u-a}{u}\, du = 2\int 1 - au^{-1}\, du = \underline{2(u - a\ln|u|) + C}.$$

Thus, $2\left(a - \sqrt{v} - a\ln|a-\sqrt{v}|\right) = \frac{k}{m}t + C.$

We need to solve for v, right?

But, you cannot do it! Thus, we need to get creative. What can we do? We can solve for t.

$$t = 2\frac{m}{k}a - 2\frac{m}{k}\sqrt{v} - \frac{m}{k}a\ln(a-\sqrt{v})^2 + C$$

Recall $a = \frac{mg}{k}$. We simplify a bit and get

$$t = -\frac{2m}{k}\sqrt{v} - \frac{m^2 g}{k^2}\ln\left(\frac{mg}{k} - \sqrt{v}\right)^2 + C \qquad \left(C = C + \frac{2ma}{k}\right)$$

Now we find C. Since $v(0) = v_0$ we have

$$0 = -\frac{2m}{k}\sqrt{v_0} - \frac{m^2 g}{k^2}\ln\left(\frac{mg}{k} - \sqrt{v_0}\right)^2 + C.$$

Thus,

$$C = \frac{2m}{k}\sqrt{v_0} + \frac{m^2 g}{k^2}\ln\left(\frac{mg}{k} - \sqrt{v_0}\right)^2.$$

Thus,

$$t = \frac{2m}{k}\left(\sqrt{v_0} - \sqrt{v}\right) + \frac{m^2 g}{k^2}\ln\left(\frac{\frac{mg}{k} - \sqrt{v_0}}{\frac{mg}{k} - \sqrt{v}}\right)^2$$

$$\boxed{t = \frac{2m}{k}\left(\sqrt{v_0} - \sqrt{v}\right) + \frac{m^2 g}{k^2}\ln\left(\frac{mg - k\sqrt{v_0}}{mg - k\sqrt{v}}\right)^2.}$$

What to do with this? Graph it.

We simplify by setting $m = k = g = 1$ and $v_0 = 0$.

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> plot(-2*sqrt(v)+ln(1/(1-sqrt(v))^2),v=0..4, color=black, thickness=4);
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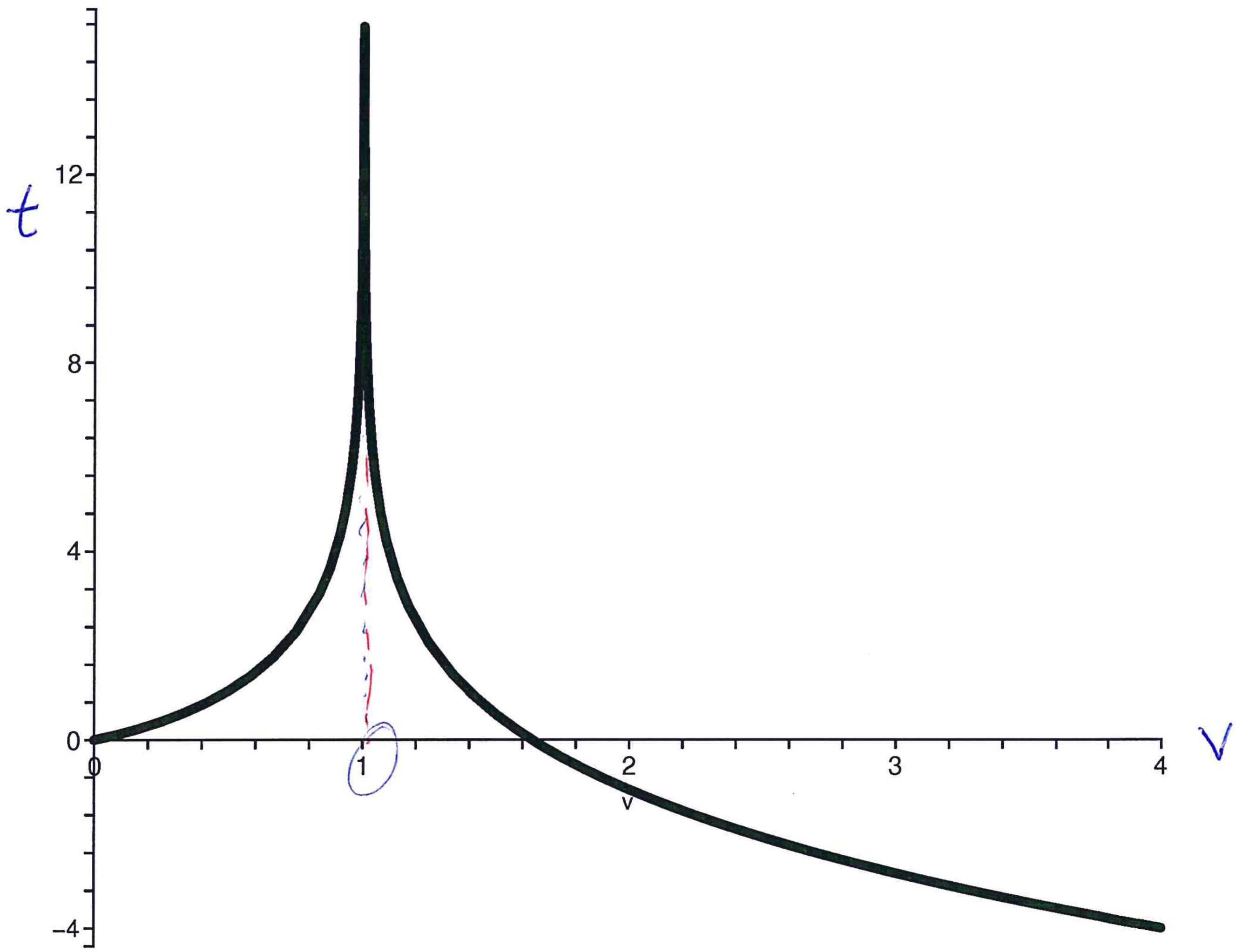
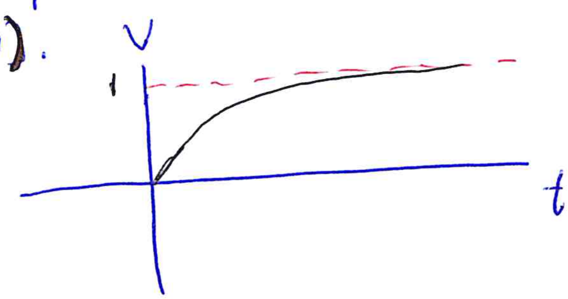


What does this mean? The terminal velocity is 1.
The graph we want is the inverse of the first branch,
$v \in [0, 1)$.



I'll run it again with $v_0 = 4$.

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> plot(-2*(sqrt(4)-sqrt(v))+ln((1-sqrt(4))^2/(1-sqrt(v))^2),v=0..4,
  color=black, thickness=4);
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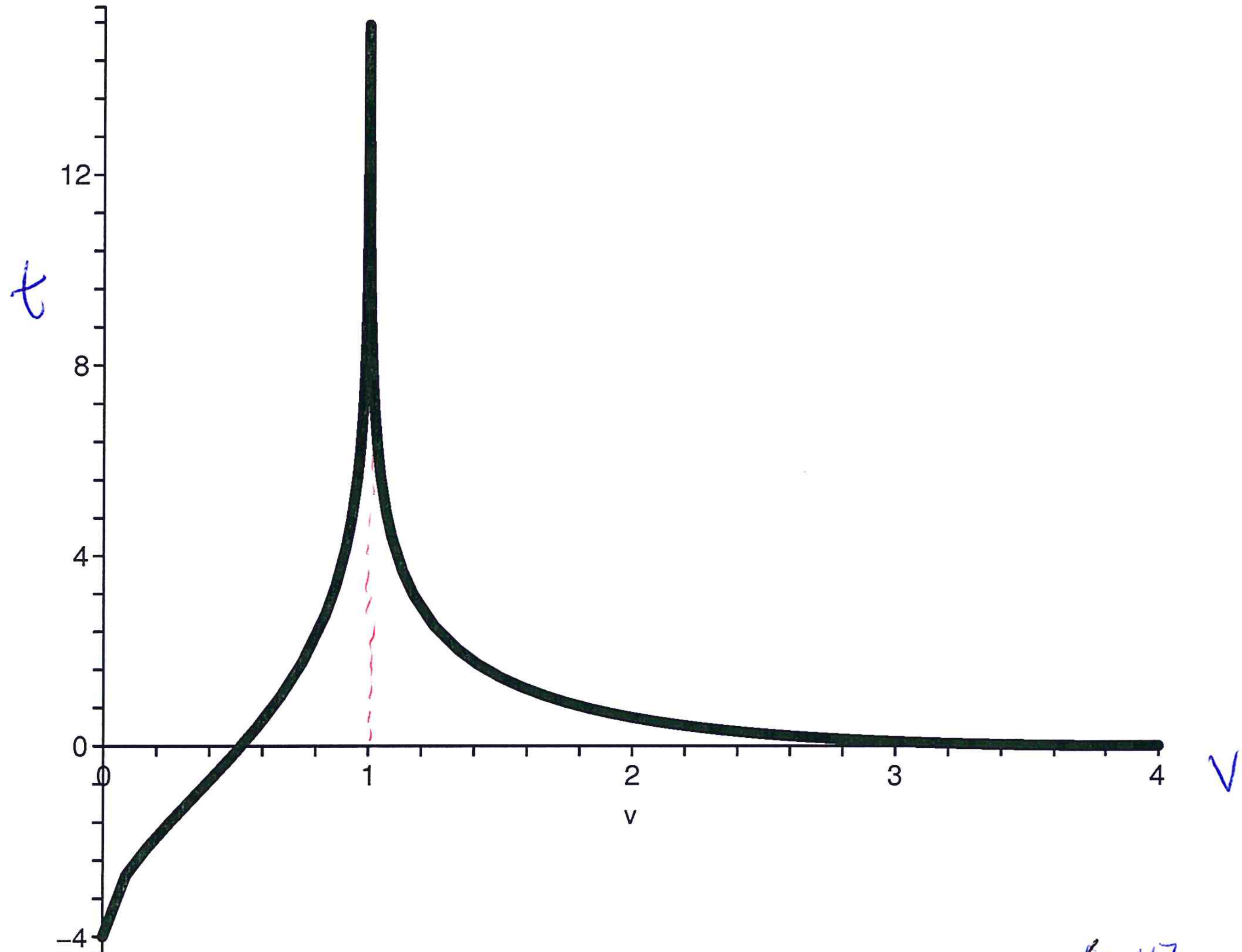
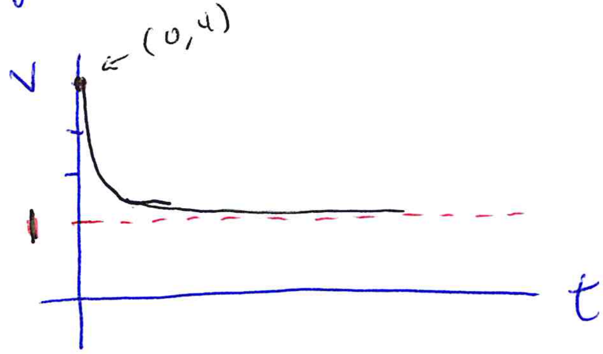


Now we are on the second branch, $v \in [1,4]$.
Our graph of v vs t is this part flipped.



Terminal velocity is still 1.

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Now that we understand how to think about the formula for t, we can find the terminal velocity for general m, k, g + V_0.

The singularity in the graphs was due to the ln term. When $mg - k\sqrt{V} \to 0$ we get $\ln(\infty) = \infty$. Therefore, the terminal velocity is

$$\boxed{V_{\text{terminal}} = \left(\frac{mg}{k}\right)^2 .}$$