3.6 Variation of Parameters.

Suppose we want to solve $y'' + py' + qy = g$ (\#) and that \{\(Y_1, Y_2\)\} is a fundamental solution set for the homogeneous problem

$$y'' + p \ y' + q \ y = 0. \quad (\#)$$

Thus \(Y_h = C_1Y_1 + C_2Y_2\) gives all solutions to \(\#\). Then a particular solution to the non homogeneous problem (\#) is given by

$$y_p = U_1(t)Y_1(t) + U_2(t)Y_2(t)$$

where

$$U_1(t) = - \int \frac{Y_2(t)g(t)}{W(Y_1, Y_2)} \ dt$$

and

$$U_2(t) = \int \frac{Y_1(t)g(t)}{W(Y_1, Y_2)} \ dt.$$ 

Then the general solution is

$$y = C_1Y_1 + C_2Y_2 + y_p.$$ 

The formulas for \(U_1\) and \(U_2\) will be derived later. First we do a couple of examples. They get messy.
Ex. Solve $y'' + y = \tan(t)$. \(-\pi < t < \pi\).

So.\(1\) We know $y_h = C_1 \sin(t) + C_2 \cos(t)$ is the general solution for $y'' + y = 0$. Let $y_1 = \sin(t)$ and $y_2 = \cos(t)$. Recall $W(y_1, y_2) = -1$. Thus

$$u_1 = -\int \frac{\cos \tan t}{-1} dt = \int \sin t dt = -\cos t.$$

$$u_2 = \int \frac{\sin t + \tan t}{-1} dt = -\int \frac{\sin t}{\cos t} dt =$$

$$\int \frac{\cos^2 t - 1}{\cos t} dt = \int \cos t - 5 \cot t dt$$

$$= \sin t - \ln(\sec t + \tan t)$$

Thus $y_p = u_1 y_1 + u_2 y_2 = -\cos t \sin t + \sin t \cos t$

$$-\cos t \ln(\sec t + \tan t)$$

$$= -\cos t \ln(\sec t + \tan t).$$

Now, we check this to be sure we did not make a mistake.

*Look up the derivation of this integral formula.*
\[
\gamma_p = -\cos \theta \ln (\sec \theta + \tan \theta)
\]
\[
\gamma_p' = \sin \theta \sec \theta (\sec \theta + \tan \theta) \cos \theta \frac{\sec \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta}
\]
\[
= \sin \theta \sec \theta (\sec \theta + \tan \theta) + 1
\]
\[
\gamma_p'' = \cos \theta \ln (\sec \theta + \tan \theta) + \sin \theta \frac{\sec \theta + \sec \theta \tan \theta}{\sec \theta + \tan \theta}
\]
\[
= \cos \theta \ln (\sec \theta + \tan \theta) + \tan \theta
\]
\[
\gamma_p'' + \gamma_p = \cos \theta \ln (\sec \theta + \tan \theta) + \tan \theta - \cos \theta \ln (\sec \theta + \tan \theta)
\]
\[
= \tan \theta. \text{ Yea!}
\]

Now, the general solution is

\[
y = C_1 \sin \theta + C_2 \cos \theta - \cos \theta \ln (\sec \theta + \tan \theta)
\]
\[
\text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.
\]
Ex \[ t^2 y'' - 2y = 3(t^2-1), \ t > 0, \ y(1) = 0, \ y'(1) = 2. \]

Sol \ First we try \( y = t^n \) for the homogeneous problem
\[ t^2 y'' - 2y = 0 \quad (\#) \]
\[ y'' = n(n-1) t^{n-2}. \ So \ (\#) \ becomes \]
\[ n(n-1) t^n - 2t^n = 0 \]
\[ (n^2 - n - 2) t^n = 0 \]
\[ (n-2)(n+1) = 0. \ \text{Let} \ n = 2, -1. \]

Then \( y_h = C_1 t^2 + C_2 t^{-1} \) is the general solution to (\#).

Put the original problem into standard form
\[ y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2}. \]

We see that the method of undetermined coefficients does not apply.

Thus, this is a job for the variation of parameters!
\( w + y_1 = 6^2 \text{ and } y_2 = 6^{-1}. \quad w(y, y_2) = -3 \text{ as you can check!} \)

\[
\begin{align*}
    u_1 &= -\int \frac{t(3-t^2)}{3} \, dt = \frac{1}{3} \int 3t^{-1} - t^{-3} \, dt = \ln t + \frac{1}{6} t^{-2} \\
    u_2 &= \int \frac{t^2(3-t^2)}{-3} \, dt = \int -t^2 + \frac{1}{3} \, dt = -\frac{1}{3} t^3 + \frac{1}{3} t \\
    y_p &= u_1 y_1 + u_2 y_2 = t^2 \ln t + \frac{1}{3} - \frac{1}{3} t^2 + \frac{1}{3} \\
    \text{Let } y_p &= t^2 \ln t + \frac{1}{3} \\
\end{align*}
\]

General solution is \( y = C_1 t^2 + C_2 t^{-1} + t^2 \ln t + \frac{1}{3}. \)

Find \( C_1 \) and \( C_2. \)

\[
\begin{align*}
    y(1) &= C_1 + C_2 + \frac{1}{3} = 0 \\
    y'(t) &= 2C_1 t - C_2 t^{-2} + 2t \ln t + \frac{t^2}{6} + 0 \\
    y'(1) &= 2C_1 - C_2 = 2 \\
\end{align*}
\]

\[
\begin{align*}
    C_1 + C_2 &= -\frac{1}{3} \\
    2C_1 - C_2 &= 2 \\
\end{align*}
\]

\[
\begin{align*}
    3C_1 &= \frac{3}{2} \quad \Rightarrow \quad C_1 = \frac{1}{2} \quad \Rightarrow \quad C_2 = -1. \\
\end{align*}
\]

Final answer is \( y(t) = \frac{1}{2} t^2 - t^{-1} + \frac{1}{2} + t^2 \ln t \).
Ex Find general solution to $y'' + 6y' + 9y = \frac{1}{t} e^{-3t}$, $t > 0$.

Sol The characteristic poly is $r^2 + 6r + 9 = (r + 3)^2$. Thus the solution to the homogeneous problem

$$y'' + 6y' + 9y = 0$$

is $y_h = C_1 e^{-3t} + C_2 t e^{-3t}$. ($y_1 = e^{-3t}$, $y_2 = te^{-3t}$)

$W(e^{-3t}, te^{-3t}) = e^{-6t}$ as you will surely check.

$$u_1 = -\int \frac{te^{-3t} + \frac{1}{t} e^{-3t}}{e^{-3t}} dt = -\int 1 dt = -t$$

$$u_2 = \int \frac{e^{-3t} + \frac{1}{t} e^{-3t}}{e^{-3t}} dt = \int \frac{1}{t} dt = \ln t$$

Let $y_p = -te^{-3t} + 6\ln t e^{-3t}$

$Ideas Ignore this. Why?$

Use $y_p = 6\ln t e^{-3t}$.

I will plug this into the original equation to test it.
\[ \gamma_p = t \ln t \, e^{-3t} \]

\[ \gamma_p' = \ln t \, e^{-3t} + e^{-3t} - 3t \ln t \, e^{-3t} \]

\[ \gamma_p'' = \frac{1}{t} \, e^{-3t} - 3 \ln t \, e^{-3t} - 3 \ln t \, e^{-3t} - 3 \ln t \, e^{-3t} - 3 \ln t \, e^{-3t} + 9 \ln t \, e^{-3t} \]

Then \( \gamma_p'' + 6 \gamma_p' + 9 \gamma_p = \)

\[ \left[ 9t \ln t + 6t + 66 - 18t \ln t + \frac{1}{t} - 3 \ln t - 3 - 3 \ln t - 3 + 9 \ln t \right] e^{-3t} \]

\[ = \frac{1}{t} \, e^{-3t} \]

\[ \checkmark \]

[Note: It took me three tries to get this one!]

Thus, the general solution is

\[ \gamma(t) = C_1 \, e^{-3t} + C_2 \, 6 \, e^{-3t} + t \ln t \, e^{-3t}. \]
Derivation of the Formulas for $u_1$ and $u_2$.

Consider $y'' + p(t)y' + q(t)y = g(t)$. Suppose

$y_1(t)$ and $y_2(t)$ are linearly independent solutions of the corresponding homogeneous problem.

Let

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Then $y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$.

We make a simplifying assumption that will turn out to be valid. We assume

$$u_1'y_1 + u_2'y_2 = 0.$$

Now

$$y_p' = u_1'y_1 + u_2'y_2.'$$

Then

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2'' + u_2y_2''.$$

Plug these into the original problem.

Oops! No primes!

$u_1'y_1 + u_1y_1' + u_2'y_2' + u_2y_2' + pu_1'y_1 + pu_2'y_2 + qu_1'y_1 + qu_2'y_2 = g$

Re-group these terms to get ...
\[ a_1 \left( y_1'' + p_1 y_1' + q_1 y_1 \right) + a_2 \left( y_2'' + p_2 y_2' + q_2 y_2 \right) + w_1 y_1' + w_2 y_2' = 0 \]

So, now we have

\[ w_1 y_1' + w_2 y_2' = 0 \quad (1) \]

We combine this with our simplifying assumption:

\[ w_1 y_1 + w_2 y_2 = 0 \quad (2) \]

Since \( y_1, y_2, y_1', y_2' \) are known, we have two equations and two unknowns: \( u_1', u_2' \).

To solve for \( u_1' \), multiply (1) by \( y_2 \) and (2) by \( y_2' \), then subtract.

\[ u_1' y_1' y_2 + \frac{u_2' y_2' y_2}{2} = g y_2 \]

\[ u_1' y_1 y_2' + \frac{u_2' y_2 y_2'}{2} = 0 \]

\[ u_1' (y_1' y_2 - y_1 y_2') = g y_2 \]

\[ u_1' = \frac{g y_2}{y_1' y_2 - y_1 y_2'} = \frac{y_2 g}{W(y_1, y_2)} \]

You show \( u_2' = \frac{y_1 g}{W(y_1, y_2)} \).