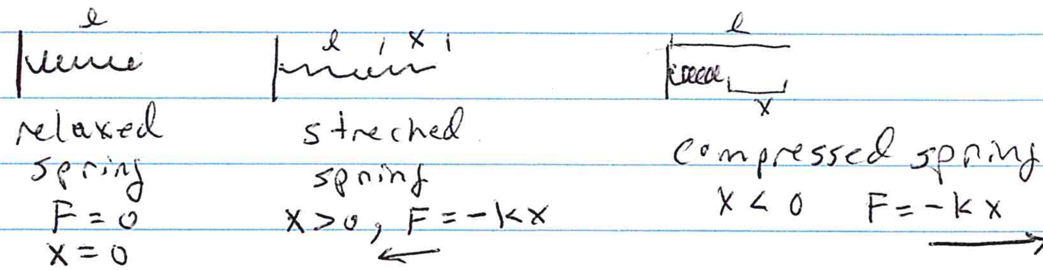
3.7 Mechanical and Electrical Vibrations

I. Mass-spring systems without damping

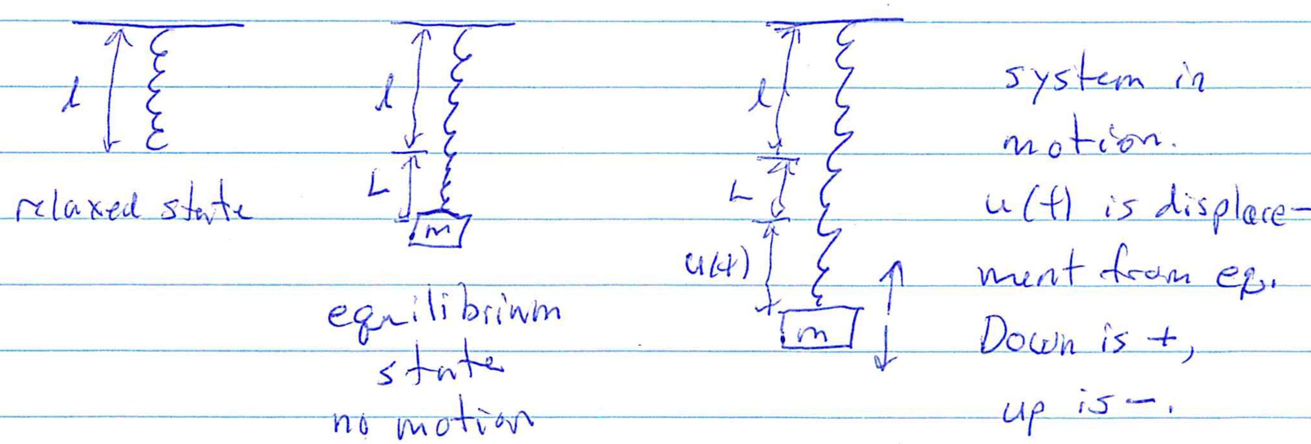
Hook's Law: $F = -kx$



k is called the spring constant.

Units for k might be Newtons/meter $= \frac{kg}{m\text{-}s^2}$ or pounds/foot $=$ slugs$/ft\text{-}s^2$.

Next we attach one end of our spring to the ceiling. Then we will attach an object of mass m to the lower end of the spring. Then we will set this system into motion.



In the equilibrium state $F = 0$. The force of the spring and the force of gravity are equal but opposite.

$$F = mg - kL = 0.$$

Thus
$$k = \frac{mg}{L}.$$

This gives us a way to measure k if we know m, or m if we know k.

If the system is in motion we analyze the forces as follows,

$$F = mg - k(L + u(t)) = \underbrace{mg - kL}_{=0} - ku$$

But
$$F = m u''(t).$$

Thus
$$m u'' = -ku \quad \text{or}$$

$$\boxed{mu'' + ku = 0.}$$

This is our model.

The characteristic equation ~~[illegible]~~

$$mr^2 + k = 0$$

has root $r = 0 \pm i\sqrt{\frac{k}{m}}$. Let $\omega_0 = \sqrt{\frac{k}{m}}$.

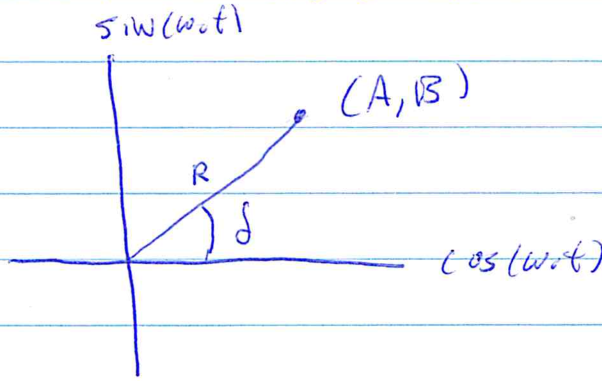
$\hookrightarrow$ natural frequency

Then the general solution is

$$\boxed{u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)}.$$

It is standard practice to put this into a more useful form using some trig.

Think "polar coordinates".



$R = \sqrt{A^2 + B^2}$

$\delta = \tan^{-1}(B/A)$ up to π.

$A = R\cos\delta$

$B = R\sin\delta$

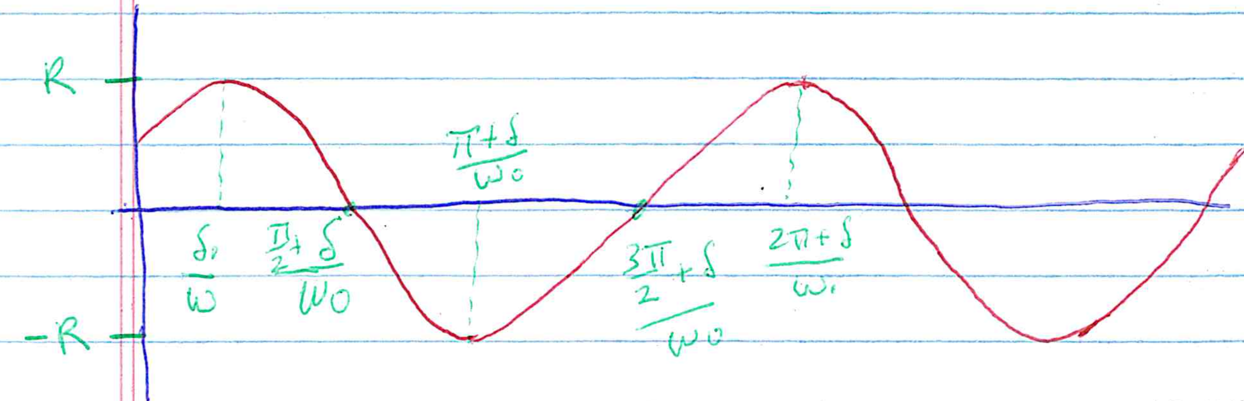
Now,

$$u(t) = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t = R\cos(\omega_0 t - \delta)$$

amplitude phase shift

Notice,

$$u(0) = A = R\cos\delta$$
$$u'(0) = \omega_0 B = R\omega_0\sin\delta$$



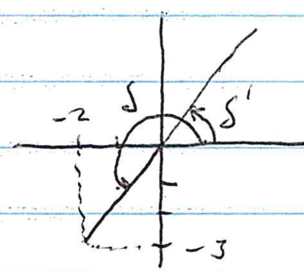
$$u(t) = R\cos(\omega_0 t - \delta)$$

Ex Suppose $u(t) = -2\cos \pi t - 3\sin \pi t$. Put into standard form.

Sol $R = \sqrt{4+9} = \sqrt{13}$. Let $\delta' = \tan^{-1}\left(\frac{-3}{-2}\right) = \tan^{-1}\left(\frac{3}{2}\right)$

$\approx 56.31^\circ$ or 0.9828 rad

But $\delta = \delta' + \pi = 4.1244$ rad or 236.31°.



Thus $\boxed{u(t) = \sqrt{13}\cos(\pi t - 4.1244)}$

II Mass-spring systems with damping

We assume there is a force resisting the motion that is proportional to the velocity.

$$F_d = -\gamma u'(t)$$

γ is called the damping constant.

I'll let you figure out what its units might be.

Now we have $F = mu'' = mg - k(L+u) - \gamma u'$, or

$$\boxed{mu'' + \gamma u' + ku = 0.} \qquad \left[\begin{array}{l}\text{Recall} \\ mg - kL = 0\end{array}\right]$$

This is our model.

The chacteristic equation is

$$mr^2 + \gamma r + k = 0.$$

Its roots are $r = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$.

Our analysis is divided into 3 cases: small damping $\gamma^2 - 4mk < 0$, large damping $\gamma^2 - 4mk > 0$ and critical damping $\gamma^2 - 4mk = 0$.

Case 1 Small damping, $\gamma^2 - 4mk < 0$.

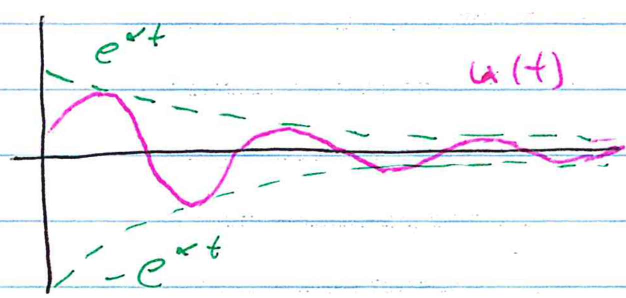
Then $r = \alpha \pm i\beta$ where $\alpha = \frac{-\gamma}{2m}$, $i\beta = \frac{\sqrt{\gamma^2 - 4mk}}{2m}$

or $\beta = \frac{\sqrt{4mk - \gamma^2}}{2m}$.

Then $u(t) = Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t$

$= e^{\alpha t}(A\cos\beta t + B\sin\beta t) = e^{\alpha t} R\cos(\beta t - \delta)$.

Since $\alpha < 0$ the graph will look like



β is called the quasi frequency.

It is less then ω_0, see page 199 eq (27).

Case 2 Large damping or overdamping, $\gamma^2 - 4mk > 0$.

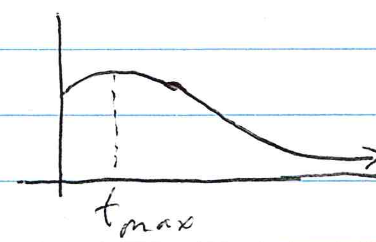
Now the roots of $mr^2 + \gamma r + k = 0$ ~~and~~ are real and negative. We have

$$u(t) = Ae^{r_1 t} + Be^{r_2 t}$$

$$u(t) \to 0 \quad \text{as } t \to \infty.$$

The initial conditions determine A and B.

Sometimes it is useful to find the maximum.



$$u' = Ar_1 e^{r_1 t} + Br_2 e^{r_2 t} = 0$$

$$Ar_1 + Br_2 e^{(r_2 - r_1)t} = 0$$

$$e^{(r_2 - r_1)t} = \frac{-Ar_1}{Br_2}$$

$$t_{max} = \frac{1}{r_2 - r_1} \ln\left(\frac{-Ar_1}{Br_2}\right).$$

Then $u(t_{max})$ = max value, if $t_{max} \geq 0$.

Case 3 Critical Damping $\gamma^2 - 4mk = 0$.

Now $r = \frac{-\gamma}{2m}$ is the only root of the characterist eq.

$$u(t) = Ae^{rt} + Bte^{rt} = Ae^{-\frac{\gamma t}{2m}} + Bte^{-\frac{\gamma t}{2m}}.$$

Show $t_{max} = -\frac{rA+B}{rB}$.

Example (3.7 #9) A mass of 20 grams stretches a spring 5 cm. Suppose $\gamma = 400$ dyne-sec/cm. If the mass is pulled down 2 cm and then released, find $u(t)$. Express your answer as a cosine wave if the damping is small. Graph it. For what time T does $t \geq T \Rightarrow |u(t)| < 0.05$ cm?

Solution Step 1. Set up model.

$m = 20$ grams $L = 5$ cm $\gamma = 400$, $u(0) = 2$, $u'(0) = 0$.

Find k. $k = \frac{mg}{L} = \frac{20 \cdot 980 \text{ cm/s}^2}{5} = 3920$.

Thus the model is

$$\boxed{20u'' + 400u' + 3920 = 0, \quad u(0) = 2, \quad u'(0) = 0.}$$

Step 2 Find general solution. The characteristic poly is

$$20r^2 + 400r + 3920 = 0$$

You can check that $r = -10 \pm i\sqrt{96}$. Thus,

$$u(t) = e^{-10t}\left(A\cos\sqrt{96}t + B\sin\sqrt{96}t\right).$$

Step 3 Find A and B. $u(0) = 2$, $u'(0) = 0$

$$u(0) = A \Rightarrow A = 2$$

$$u'(t) = -10e^{-10t}\left(\qquad\qquad\right) + e^{-10t}\left(-A\sqrt{96}\sin\sqrt{96}t + B\sqrt{96}\sin\sqrt{96}t\right)$$

$$\begin{aligned} u'(0) &= -10(A) + (B\sqrt{96}) \\ &= -20 + B\sqrt{96} \\ \Rightarrow B &= \frac{20}{\sqrt{96}} \approx 2.041241\overset{4}{\cancel{5}}523 \end{aligned}$$

Step 4 Convert to cosine wave.

$$R = \sqrt{A^2 + B^2} = \frac{7}{\sqrt{6}} \approx 2.8577380333$$

$$\delta = \arctan\left(\frac{B}{A}\right) \approx 0.79560295355 \text{ radians.}$$ | no need to add π

Thus $$\boxed{u(t) \approx 2.858\, e^{-10t}\cos\left(\sqrt{96}\,t - 0.7956\right)}$$

Plots for 3.7 #9
using Maple

```
> R:=7/sqrt(6);
```

$$R := \frac{7}{6}\sqrt{6}$$

```
> delta:=evalf(arctan(20/sqrt(96),2));
```

$$\delta := 0.7956029534$$

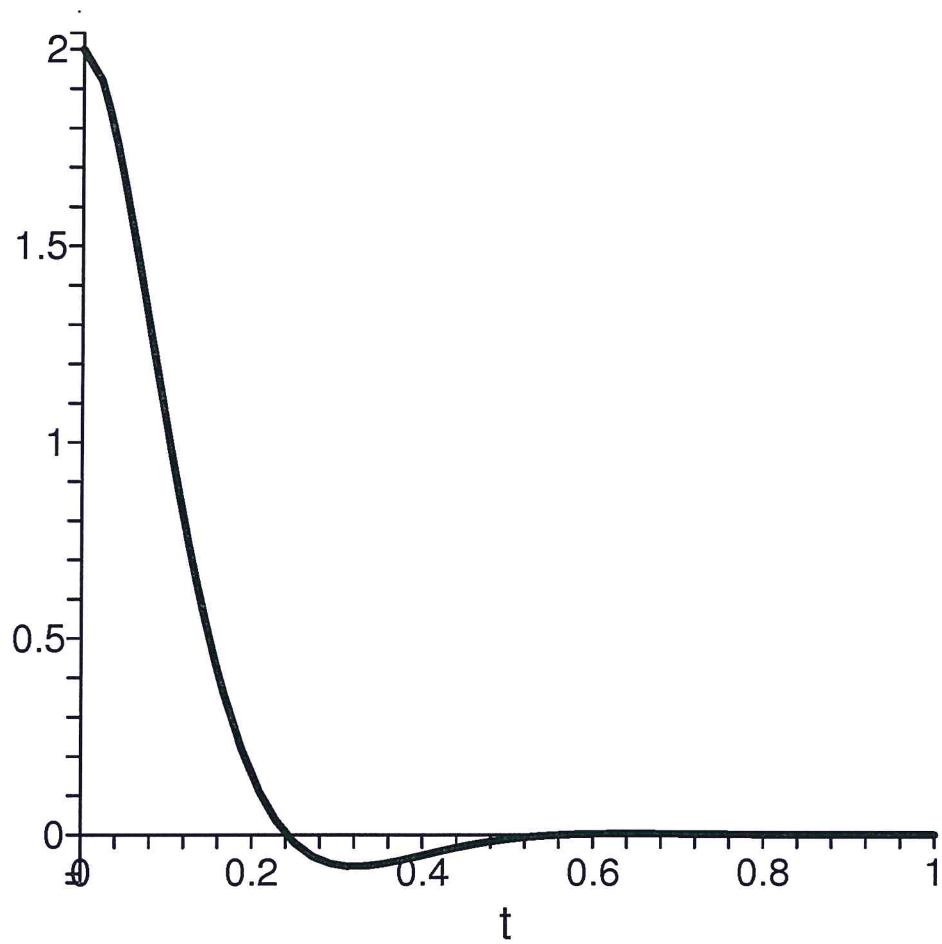
```
> beta:=sqrt(96);
```

$$\beta := 4\sqrt{6}$$

```
> alpha:=-10;
```

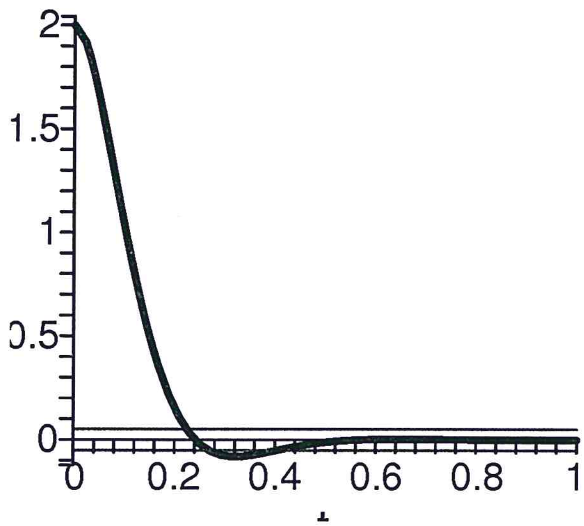
$$\alpha := -10$$

```
> plot(R*exp(alpha*t)*cos(beta*t - delta),t=0..1,color=black,thickness=2);
```



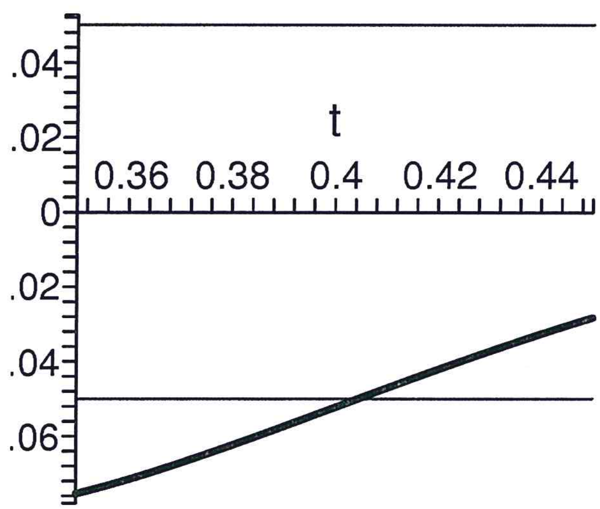
Find T for which t>T implies |u(t)| < 0.05

```
> plot([-0.05,R*exp(alpha*t)*cos(beta*t -
  delta),0.05],t=0..1,thickness=[1,2,1],color=[red,black,red]);
```



We zoom in for a better look.

```
> plot([-0.05,R*exp(alpha*t)*cos(beta*t -
  delta),0.05],t=0.35..0.45,thickness=[1,2,1],color=[red,black,red]);
```



T = 0.4045, approximately.

We can get a better approximation using the solve command.

```
> solve(R*exp(alpha*t)*cos(beta*t - delta) = -0.05, t);
```

0.4045411987

Example (3.7 #11) A spring is stretched 10 cm by a force of 3N (Newtons). A mass of 2 kg is hung from the spring. When the velocity of the mass is 5 m/s a damping force of 3 N is measured. Now suppose the mass is pulled down 5 cm below equilibrium and given an initial velocity of 10 cm/s downward. Find a formula for the motion of this mass-spring system and graph it.

Solution Step 1, Set up the model.

$m = 2$

$\gamma = 3/5$

convert to meters!

$k = 3/\underline{0.1} = 30$

$u(0) = 5\text{ cm} = 0.05\text{ m}$

$u'(0) = 10\text{ cm/s} = 0.1\text{ m/s}$

Thus our model is

$$\boxed{2u'' + \tfrac{3}{5}u' + 30u = 0 \quad u(0) = 0.05,\ u'(0) = 0.1.}$$

Step 2 Find gen. sol. I multiplied through by 5 so as not to have fractions.

$$10u'' + 3u' + 150u = 0$$

$$10r^2 + 3r + 150 = 0$$

$$r = \frac{-3 \pm \sqrt{9 - 40 \cdot 150}}{20} = -\frac{3}{20} \pm \frac{i\sqrt{5991}}{20} \approx$$

$-0.15 \pm i\,3.8700775$

Thus

$$u(t) = e^{-\frac{3}{20}t}\left(A\cos\left(\frac{\sqrt{5991}}{20}t\right) + B\sin\left(\frac{\sqrt{5991}}{20}t\right)\right)$$

Step 3 Find A and B. $u(0) = 0.05$, $u'(0) = 0.1$

$$u(0) = A \Rightarrow A = 0.05$$

$$u'(t) = -\frac{3}{20}e^{-\frac{3}{20}t}\left(\qquad\qquad\right) + e^{-3/20t}\left(-A\frac{\sqrt{\ }}{20}\sin(\ ) + B\frac{\sqrt{\ }}{20}\cos(\ )\right)$$

$$u'(0) = -\frac{3}{20}A + B\frac{\sqrt{5991}}{20} = -0.0075 + B\frac{\sqrt{5991}}{20} = 0.1$$

$$\Rightarrow B = \frac{20}{\sqrt{5991}}(0.1075) \approx 0.02777722138$$

Step 4 Convert to cosine wave.

$$R = \sqrt{A^2 + B^2} \approx 0.057197675$$

$$\delta = \arctan\left(\frac{B}{A}\right) \approx 0.50709 \text{ rad.}$$ (No need to add π)

$$\boxed{u(t) \approx (0.057)e^{-0.15t}\cos(3.870t - 0.50709)}$$

Plot for 3.7 #11 using Maple

```
> R:=sqrt((0.05)^2 + (2.15)^2/5991);
```

$$R := 0.05719767502$$

```
> delta:=evalf(arctan(43/sqrt(5991)));
```

$$\delta := 0.5070900009$$

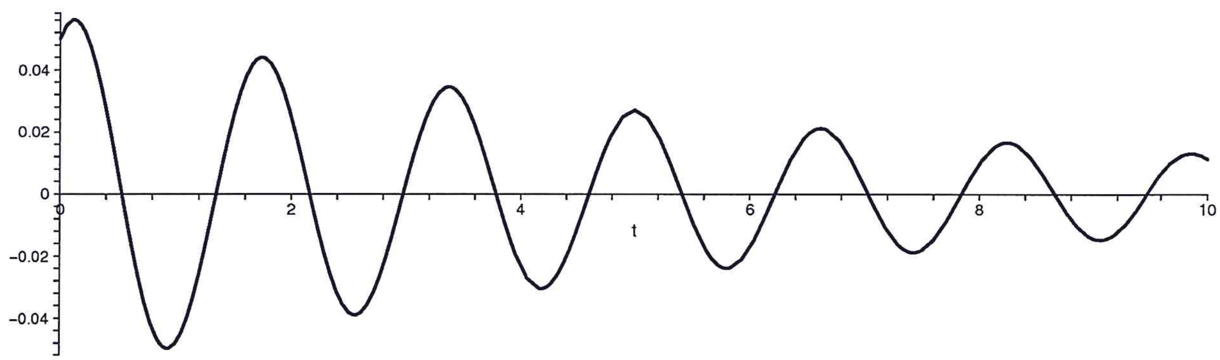
```
> beta:=sqrt(5991)/20;
```

$$\beta := \frac{1}{20}\sqrt{5991}$$

```
> alpha:=-3/20;
```

$$\alpha := \frac{-3}{20}$$

```
> plot(R*exp(alpha*t)*cos(beta*t - delta),t=0..10,color=black,thickness=2);
```

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>
```

Example (3.7 #21) **The Logarithmic Decrement**

(a) For the damped oscillation described by Eq. (26), show that the time between successive maxima is

$$T_d = 2\pi/\mu.$$

(b) Show that the ratio of the displacements at two successive maxima is given by $\exp(\gamma T_d/2m)$. Observe that this ratio does not depend on which pair of maxima is chosen. The natural log of this ratio is called the logarithmic decrement and is denoted by Δ.

Solution (a) Eq (26) is

$$u(t) = R e^{\frac{-\gamma t}{2m}} \cos(\mu t - \delta).$$

We are pretty sure T_d is just the period of $\cos(\mu t - \delta)$ which is $2\pi/\mu$, but we check.

Set $\frac{du}{dt} = 0$ and solve for t.

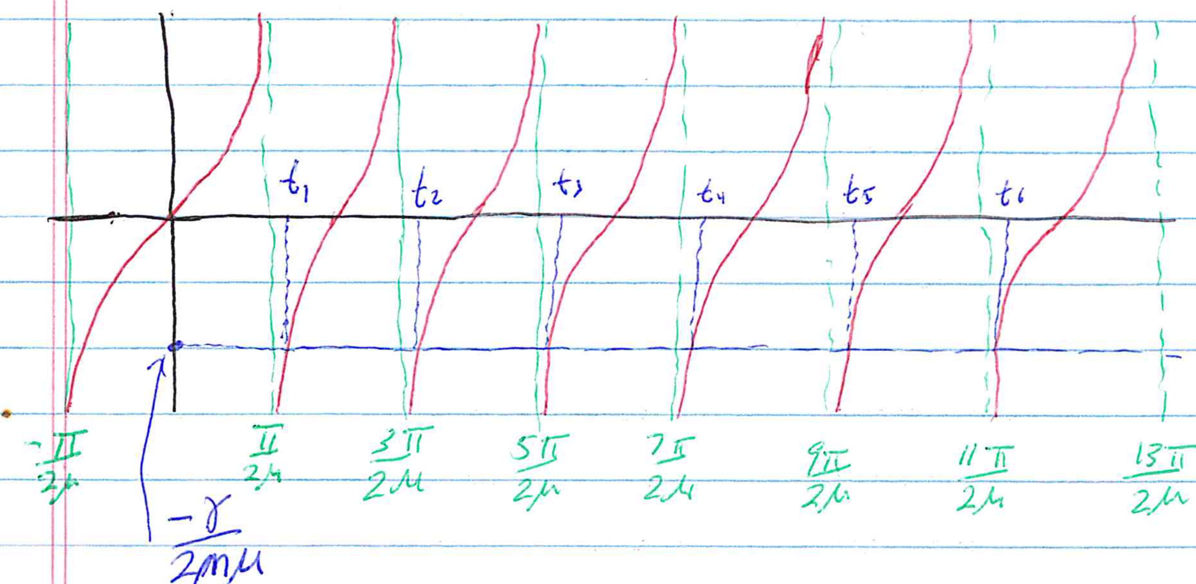
$$u'(t) = R \frac{-\gamma t}{2m} e^{\frac{-\gamma t}{2m}} \cos(\mu t - \delta) - R\mu e^{\frac{-\gamma t}{2m}} \sin(\mu t - \delta) = 0.$$

Thus $\tan(\mu t - \delta) = \frac{-\gamma}{2m\mu}$ (As you will check!)

period of the tangent function

Let $t_n = \overbrace{\frac{1}{\mu}\tan^{-1}\left(\frac{-\gamma}{2m\mu}\right) + \frac{\delta}{\mu}}^{=X} + n\frac{\pi}{\mu}$, $n = 1, 2, 3, \dots$

See the graph below to understand the idea ($\delta = 0$ for simplicity.)



Every other t_n corresponds to a maxima. Thus

$$T_d = t_{n+2} - t_n = \left(X + \frac{(n+2)\pi}{\mu}\right) - \left(X + \frac{n\pi}{\mu}\right) = \frac{2\pi}{\mu}$$

as we thought!

(b) We need to find $\dfrac{u(t_n)}{u(t_{n+2})}$.

We know $t_{n+2} = t_n + \frac{2\pi}{\mu}$.

$$u(t_n) = R e^{\frac{-\gamma t_n}{2m}} \cos(\mu t_n - \delta)$$

$$u(t_{n+2}) = u\left(t_n + \frac{2\pi}{\mu}\right) = R e^{\frac{-\gamma t_n}{2m} - \frac{\cancel{2}\gamma\pi}{\cancel{2}m\mu}} \cos\left(\mu\left(t_n + \frac{2\pi}{\mu}\right) - \delta\right)$$

$$= R e^{\frac{-\gamma t_n}{2m}} e^{-\frac{\gamma\pi}{m\mu}} \cos(\mu t_n + 2\pi - \delta)$$

$$= \text{''} \quad \text{''} \quad \text{''} \quad \cos(\mu t_n - \delta).$$

$$\frac{u(t_n)}{u(t_{n+2})} = \frac{\cancel{R}\, \cancel{e^{\frac{-\gamma t_n}{2m}}}\, \cancel{\cos(\mu t_n - \delta)}}{\cancel{R}\, \cancel{e^{\frac{-\gamma t_n}{2m}}}\, e^{\frac{-\gamma\pi}{m\mu}}\, \cancel{\cos(\mu t_n - \delta)}} = e^{\frac{\gamma\pi}{m\mu}}$$

Since $T_d = \frac{2\pi}{\mu}$, $e^{\frac{\gamma\pi}{m\mu}} = \boxed{e^{\frac{\gamma T_d}{2m}}}$ Victory!!

Clearly the result is independent of n.

Let $\Delta = \ln\left(e^{\frac{\gamma T_d}{2m}}\right) = \frac{\gamma T_d}{2m}$.

Then $\gamma = \frac{2m\Delta}{T_d}$. Each of m, Δ and T_d is easy to measure in an experiment. Thus we have an experimental means of find the damping constant. That is the point of this problem.