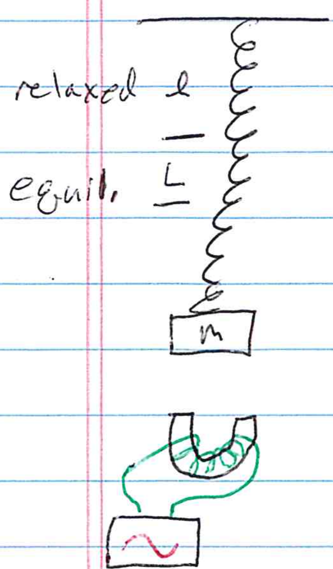
3.8 Forced Vibrations



Suppose there is an external force opperating on the object at the end of our spring. Prehaps an electromagnet ~~exto~~ exerts an oscillating force from below. Call this force $F_e(t)$.

We analyze the forces.

$$F = \underbrace{mg}_{\text{gravity}} - \underbrace{k(L+u(t))}_{\text{spring force}} - \underbrace{\gamma u'(t)}_{\text{resistance}} + \underbrace{F_e(t)}_{\text{external}}$$

Since $F = mu''(t)$ we have

$$m u'' = \underbrace{mg - kL}_{=0} - ku - \gamma u' + F_e$$

$$\boxed{m u'' + \gamma u' + k u = F_e}$$

For now, we are going to study this when

$$\boxed{F_e = F_0 \cos(\omega t)}$$

First we assume $\gamma = 0$, that is no damping. Then we have

$$mu'' + ku = F_0 \cos(\omega t)$$

Recall $\omega_0 = \sqrt{\frac{k}{m}}$. We know the solution to the homogeneous eq. ~~is~~, $mu'' + ku = 0$, is

$$u_h = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t).$$

Case 1 Assume $\omega_0 \neq \omega$. Then our particular solution is of the form,

$$u_p = A \cos(\omega t) + B \sin(\omega t).$$

We need to find A and B.

$$u_p'' = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)$$

Thus

$$mu_p'' + ku_p = -mA\omega^2 \cos(\omega t) - mB\omega^2 \sin(\omega t) + kA\cos(\omega t) + kB\sin(\omega t)$$

$$= A(k - m\omega^2)\cos(\omega t) + B(k - m\omega^2)\sin(\omega t)$$

This must $= F_0 \cos(\omega t)$

Therefore $A = \dfrac{F_0}{k - m\omega^2}$ and $B = 0$.

We rewrite A as follows

$$A = \frac{F_0}{k - m\omega^2} = \frac{F_0}{m\left(\frac{k}{m} - \omega^2\right)} = \boxed{\frac{F_0}{m(\omega_0^2 - \omega^2)}}$$

Notice this solution would fail if $\omega_0 = \omega$ and that when ω_0 and ω are close A gets really big.

For the sake of simplicity let's suppose

$$u(0) = u'(0) = 0.$$

Then you can check that

$$C_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad C_2 = 0.$$

Thus,

$$\boxed{u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos(\omega t) - \cos(\omega_0 t)\right)}.$$

Next we will use some trig to rewrite in a form that is handy.

Recall:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B.$$

Thus,

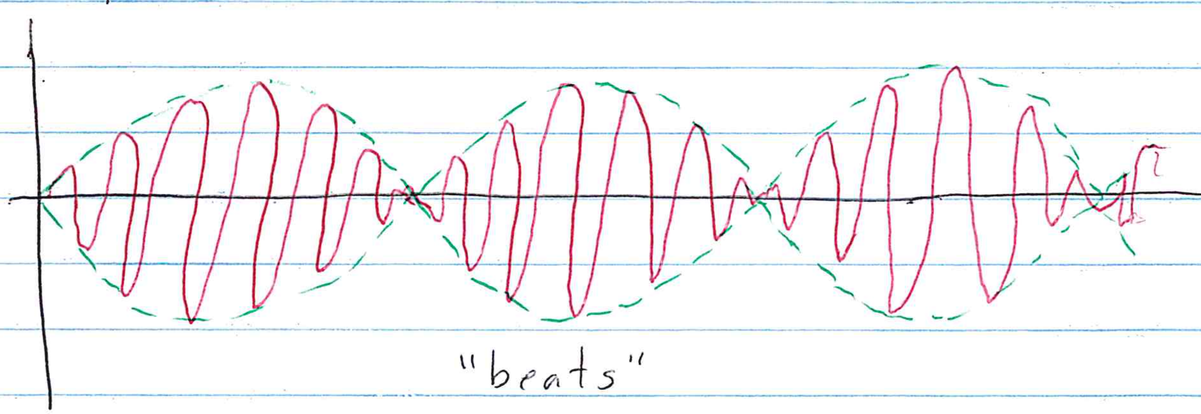
$$\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$$

Let $A = \frac{\omega_0 t + \omega t}{2}$ and $B = \frac{\omega_0 t - \omega t}{2}$.

Then $A + B = \omega_0 t$ and $A - B = \omega t$.

$$\boxed{\text{Now} \quad u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 - \omega}{2} t\right) \sin\left(\frac{\omega_0 + \omega}{2} t\right)}$$

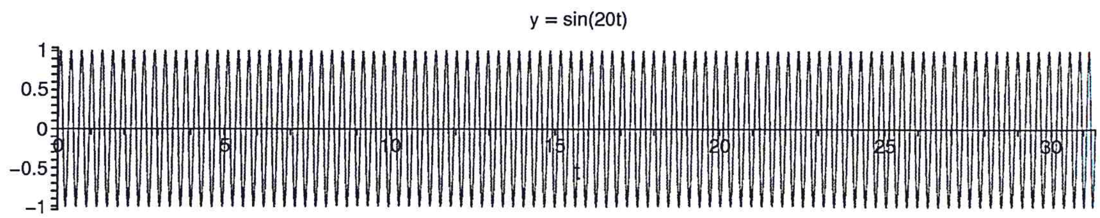
If ω_0 is close to ω, then $\frac{\omega_0 - \omega}{2}$ is small compared to $(\omega_0 + \omega)/2$. Then the graph will look like



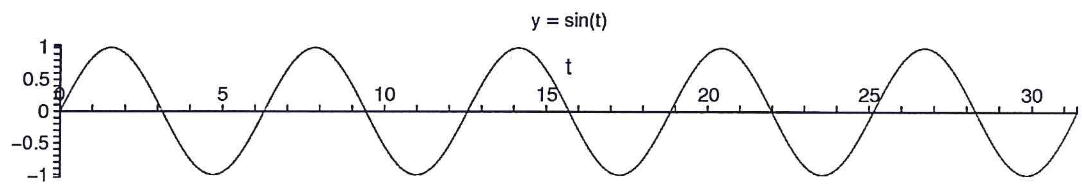
"beats"

The beat goes on ....

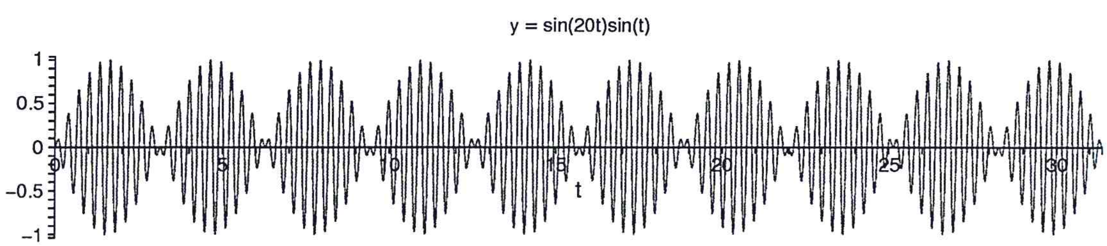
```
> plot(sin(20*t),t=0..10*Pi,numpoints=1000, title="y = sin(20t)");
>
```




```
> plot(sin(t),t=0..10*Pi,numpoints=1000, title="y = sin(t)");
>
```




```
> plot(sin(t)*sin(20*t),t=0..10*Pi,numpoints=1000,title="y =
  sin(20t)sin(t)");
```



Case 2 $\quad \omega = \omega_0$

Now let $u_p = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$

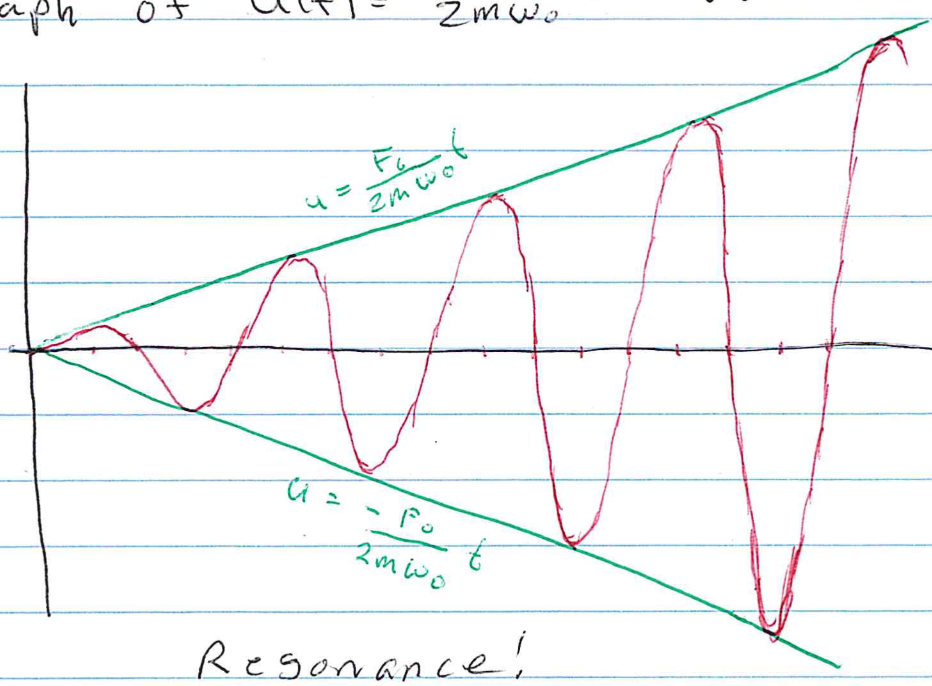
Plug into $mu_p'' + ku_p = F_0\cos(\omega t)$ and you should get

$$A = 0, \quad B = \frac{F_0}{2m\omega_0}.$$

Thus, $\quad u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$

If we set $u(0) = u'(0) = 0$ then you should get $C_1 = C_2 = 0$.

The graph of $u(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$ is



Resonance!

Forced Vibrations with Damping ($\gamma > 0$).

We consider $mu'' + \gamma u' + ku = F_e(t) = F_0 \cos(\omega t)$. (*)

The solution of the homogeneous problem

$mu'' + \gamma u' + ku = 0$ is

$$u_h = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \left(r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} \text{ both real and } < 0 \right), \\ C_1 e^{rt} + C_2 t e^{rt} & \left(r = -\gamma/2m < 0 \right), \\ \text{or } e^{-\gamma t/2m}(C_1 \cos \beta t + C_2 \sin \beta t), & \left(\beta = \sqrt{4km - \gamma^2}/2m \right). \end{cases}$$

In all three cases $\lim_{t\to\infty} u_h(t) = 0$. Because of this u_h is call the transient part of the solution.

The particular solution to (*) is of the form

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t).$$

It is called the stead state part of the solution.

We will study how $u_p(t)$ changes with different values of ω.

We can rewrite $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$ in the form $R\cos(\omega t - \delta)$. Later, we will derive these formulas

$$R = \frac{F_0}{\Delta} \qquad \cos\delta = \frac{m(\omega_0^2-\omega^2)}{\Delta} \qquad \sin\delta = \frac{\gamma\omega}{\Delta} \qquad (\star)$$

where $\Delta = \sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$ and $\omega_0 = \sqrt{\frac{k}{m}}$.

This system no longer blows up when $\omega_0 = \omega$ but R does hits its maximum when regarded as a function of ω where $\omega = \omega_0$ you can check,

$$\left.\frac{dR}{d\omega}\right|_{\omega_0} = 0 \quad \text{and} \quad R(\omega_0) = \frac{F_0}{\gamma\omega_0}.$$

(See graphs on page 211.)

Here is the derivation of the formulas (*).

$$u_p = A\cos\omega t + B\sin\omega t$$
$$u_p' = -A\omega\sin\omega t + B\omega\cos\omega t$$
$$u_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$m u_p'' + \gamma u_p' + k u_p =$

$$(-mA\omega^2 + \gamma B\omega + kA)\cos\omega t + (-mB\omega^2 - \gamma A\omega + kB)\sin\omega t$$

must $= F_0\cos(\omega t)$. Thus

$$(k - m\omega^2)A + \gamma\omega B = F_0 \quad \text{and}$$

$$-\gamma\omega A + (k - m\omega^2)B = 0.$$

$$\begin{aligned} \gamma\omega(k-m\omega^2)A + \gamma^2\omega^2 B &= \gamma\omega F_0 \\ -\gamma\omega(k-m\omega^2)A + (k-m\omega^2)^2 B &= 0 \end{aligned}$$

$$[\gamma^2\omega^2 + (k-m\omega^2)^2]B = \gamma\omega F_0$$

$$B = \frac{\gamma\omega F_0}{\gamma^2\omega^2 + (k-m\omega^2)^2}$$

$$A = \frac{(k-m\omega^2)B}{\gamma\omega}.$$

Next we find $R = \sqrt{A^2+B^2}$.

$$R^2 = \frac{(k-\omega^2 m)^2}{\gamma^2\omega^2}B^2 + B^2 = \left[\frac{(k-\omega^2 m)^2}{\gamma^2\omega^2} + 1\right]B^2$$

$$= \left[\frac{\cancel{(k-\omega^2 m)^2 + \gamma^2\omega^2}}{\cancel{\gamma^2 \omega^2}}\right]\frac{\cancel{\gamma^2\omega^2} F_0^2}{(\gamma^2\omega^2 + (k-\omega^2 m)^2)^{\cancel{2}}}$$

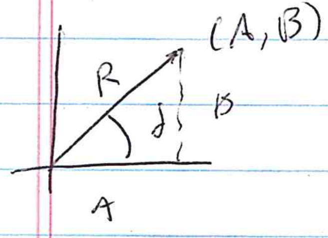
$$= \frac{F_0^2}{\gamma^2\omega^2 + (k-\omega^2 m)^2}$$

Now, $(k - m\omega^2) = m\left(\frac{k}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2)$.

Let $\Delta = \left(\gamma^2\omega^2 + m^2(\omega_0^2-\omega^2)^2\right)^{1/2}$. Then

$$R^2 = \frac{F_0^2}{\Delta^2} \quad \text{so} \quad \boxed{R = \frac{F_0}{\Delta}}.$$

Now for δ.



$$\sin\delta = \frac{B}{R} = \frac{\frac{\gamma\omega F_0}{\Delta^2}}{\frac{F_0}{\Delta}} = \frac{\gamma\omega}{\Delta}$$

$$\cos\delta = \frac{A}{R} = \frac{\frac{m(\omega_0^2-\omega^2)}{\gamma\omega}\cdot B}{\frac{F_0}{\Delta}} =$$

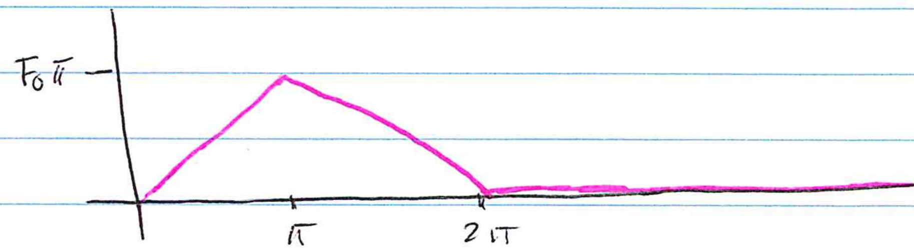
$$\frac{m(\omega_0^2-\omega^2)}{\gamma\omega}\cdot\frac{B}{R} = \frac{m(\omega_0^2-\omega^2)}{\Delta}.$$

Example (3.8 #15) Find a differentiable solution to

$$u'' + u = F(t), \quad u(0) = 0, \quad u'(0) = 0 \quad \text{where}$$

$$F(t) = \begin{cases} F_0 t & 0 \le t < \pi \\ F_0(2\pi - t) & \pi \le t < 2\pi \\ 0 & 2\pi \le t \end{cases}$$

Solution First, I alway like to graph the forcing func.



We will divide the problem into three parts

① $t \in [0, \pi]$, ② $t \in [\pi, 2\pi]$, ③ $t \geq 2\pi$.

For ① we find a solution using $u(0) = u'(0) = 0$. Call this solution $u_1(t)$.

For ② we find a solution, call it $u_2(t)$, using

$$u_2(\pi) = u_1(\pi), \quad u_2'(\pi) = u_1'(\pi).$$

For ③ we find a solution, call it $u_3(t)$, using

$$u_3(2\pi) = u_2(2\pi), \quad u_3'(2\pi) = u_2'(2\pi).$$

Then our solution to the original problem is

$$u(t) = \begin{cases} u_1(t) & t \in [0, \pi) \\ u_2(t) & t \in [\pi, 2\pi) \\ u_2(t) & t \geq 2\pi \end{cases}$$

Got it?

Part 1 $t \in [0, \pi)$ $\quad u'' + u = F_0 t \quad u(0) = 0 \quad u'(0) = 0.$

$$u_h = C_1 \cos t + C_2 \sin t \qquad u_p = F_0 t.$$

The general solution is

$$u_1 = C_1 \cos t + C_2 \sin t + F_0 t.$$

$$u_1(0) = C_1 \Rightarrow C_1 = 0$$

$$u_1'(t) = C_2 \cos t + F_0$$

$$u_1'(0) = C_2 + F_0 \Rightarrow C_2 = -F_0.$$

Thus

$$\boxed{u_1(t) = -F_0 \sin t + F_0 t.}$$

Part 2 $t \in [\pi, 2\pi)$

$$u_h = C_1 \cos t + C_2 \sin t \qquad u_p = F_0(2\pi - t)$$

$$u_2(t) = C_1 \cos t + C_2 \sin t + F_0(2\pi - t)$$

$$\cancel{u_2(0) = C_1 + F_0 2\pi \quad u_1}$$

$$u_2(\pi) = -C_1 + F_0 \pi \qquad u_1(\pi) = F_0 \pi$$

Thus $C_1 = 0$.

$$u_2'(t) = C_2 \cos t - F_0 \qquad u_1'(t) = -F_0 \cos t + F_0$$

$$u_2'(0) = -C_2 - F_0 \qquad u_1'(\cancel{0}\ \pi) = F_0 + F_0 = 2F_0.$$

Thus $C_2 = -3F_0$

Thus we get $\boxed{u_2(t) = -3F_0 \sin t + F_0(2\pi - t).}$

Part 3 $t \geq 2\pi$. Since the forcing func. is now zero

$$u_3(t) = C_1 \cos t + C_2 \sin t.$$

$$u_3(2\pi) = C_1 \qquad u_2(2\pi) = 0$$

Thus $C_1 = 0$

$$u_3'(t) = C_2 \cos t \qquad u_2'(t) = -3F_0 \cos t - F_0$$

$$u_3'(2\pi) = C_2 \qquad u_2'(2\pi) = -4F_0$$

Thus $C_2 = -4F_0$

$$u_3(t) = -4F_0 \sin t$$

Part 4 We put it all together to get

$$u(t) = \begin{cases} -F_0 \sin t + F_0 t & t \in [0, \pi) \\ -3F_0 \sin t + F_0(2\pi - t) & t \in [\pi, 2\pi) \\ -4F_0 \sin t & t \in [2\pi, \infty) \end{cases}$$

See next page for the graphs.

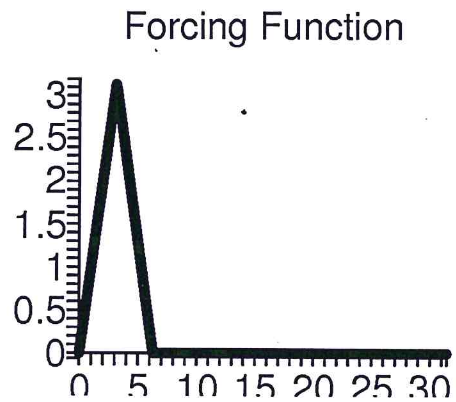
(using Maple)

Some plots for Example in class (3.8 #15)

```
> F:=t -> piecewise(t>0 and t<Pi, t, t>=Pi and t <2*Pi, 2*Pi- t); #use help to lookup "piecewise".
```

$$F := t \rightarrow piecewise(0 < t \textbf{ and } t < \pi, t, \pi \leq t \textbf{ and } t < 2\pi, 2\pi - t)$$

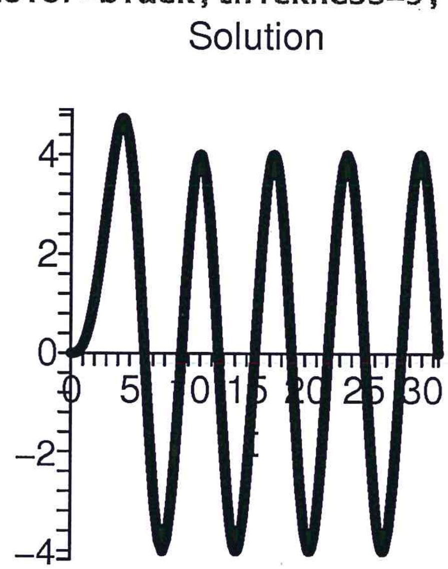
```
> plot(F(t), t=0..10*Pi,color=black,thickness=3,title=`Forcing Function`);
```




```
> u:= t -> piecewise(t>0 and t<Pi,-sin(t) + t,t>=Pi and t<=2*Pi,-3*sin(t)+2*Pi - t,t>2*Pi,-4*sin(t));
```

$$u := t \rightarrow piecewise(0 < t \textbf{ and } t < \pi, -\sin(t) + t, \pi \leq t \textbf{ and } t \leq 2\pi, -3\sin(t) + 2\pi - t, 2\pi < t, -4\sin(t))$$

```
> plot(u(t),t=0..10*Pi,color=black,thickness=3,title=`Solution`);
```




```
> evalf(2*Pi-solve(diff(-3*sin(t)+2*Pi-t,t)=0));
```

4.372552071

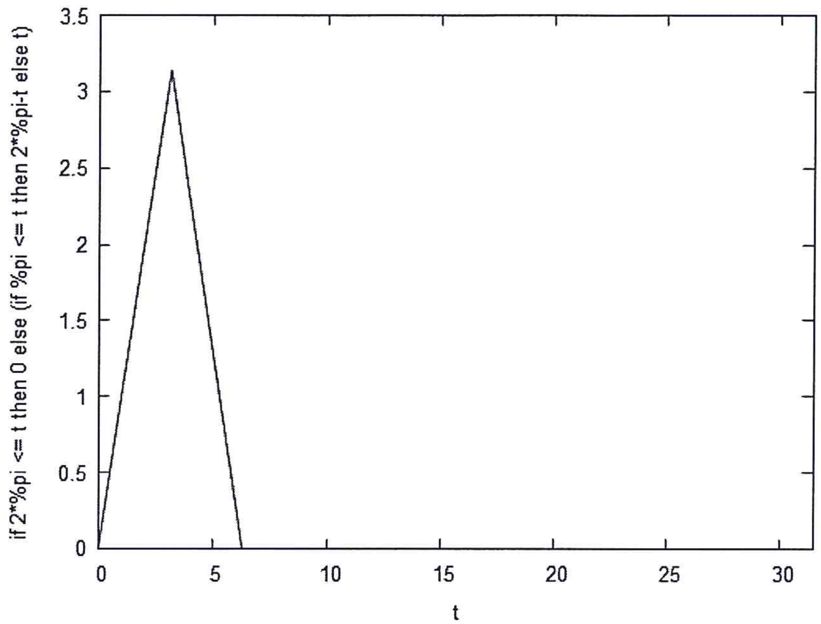
```
> evalf(u(%));
```

4.739060362

3.8 #15 graphs using Maxima

(dbm:7) F(t):= if 2*%pi<=t then 0 else if %pi<=t then 2*%pi-t else t $

(dbm:7) wxplot2d(F(t),[t,0,10*%pi])$



(dbm:7) u(t):= if t>=2*%pi then -4*sin(t) else if t>=%pi then -3*sin(t) - t +2*%pi else -sin(t) +t$

(dbm:7) wxplot2d(u(t),[t,0,10*%pi])$