3.8 Forced Vibrations

Suppose there is an external force operating on the object at the end of our spring. Perhaps an electromagnetic device exerts an oscillating force from below. Call this force $F_e(t)$.

We analyze the forces.

$$F = mg = k (L + u(t)) - y u'(t) + F_e(t)$$

*gravity*  *spring force*  *resistance*  *external*

Since $F = m u''(t)$ we have

$$m u'' = mg - kL - Ku - yu' + F_e$$

$$m u'' + y u' + Ku = F_e$$

For now, we are going to study this when

$$F_e = F_o \cos(\omega t)$$
First we assume $\gamma = 0$, that is nodamping. Then we have

$$m u'' + ku = F_0 \cos(\omega t)$$

Recall $\omega_0 = \sqrt{\frac{k}{m}}$. We know the solution to the homogeneous eq. \( m u'' + ku = 0 \) is

$$u_h = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t).$$

**Case 1** Assume $\omega_0 \neq \omega$. Then our particular solution is of the form

$$u_p = A \cos(\omega t) + B \sin(\omega t).$$

We need to find $A$ and $B$.

$$u''' = -Aw^2 \cos(\omega t) - Bw^2 \sin(\omega t)$$

Thus $m u'' + ku = -mA w^2 \cos(\omega t) - mBw^2 \sin(\omega t) + kA \cos(\omega t) + kB \sin(\omega t)$

$$= A(k - m\omega^2) \cos(\omega t) + B(k - m\omega^2) \sin(\omega t)$$

This must = $F_0 \cos(\omega t)$
Therefore \( A = \frac{F_0}{k - m \omega^2} \) and \( B = 0 \).

We rewrite \( A \) as follows

\[
A = \frac{F_0}{k - m \omega^2} = \frac{F_0}{m \left( \frac{k}{m} - \omega^2 \right)} = \frac{F_0}{m (\omega_0^2 - \omega^2)}
\]

Notice this solution would fail if \( \omega_0 = \omega \) and that when \( \omega \) and \( \omega_0 \) are close \( A \) gets really big.

For the sake of simplicity let's suppose

\[
U(0) = U'(0) = 0,
\]

Then you can check that

\[
C_1 = \frac{-F_0}{m (\omega_0^2 - \omega^2)} \quad \text{and} \quad C_2 = 0.
\]

Thus

\[
U(t) = \frac{F_0}{m (\omega_0^2 - \omega^2)} \left( \cos(\omega t) - \cos(\omega_0 t) \right).
\]

Next we will use some trig to rewrite in a form that is handy,
Recall:
\[ \cos(A+B) = \cos A \cos B - \sin A \sin B, \]
\[ \cos(A - B) = \cos A \cos B + \sin A \sin B. \]

Thus,
\[ \cos(A-B) - \cos(A+B) = 2 \sin A \sin B \]

Let \( A = \frac{\omega_0 t + \omega t}{2} \) and \( B = \frac{\omega_0 t - \omega t}{2} \).

Then \( A + B = \omega_0 t \) and \( A - B = \omega t \).

Now \( u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \left( \frac{\omega_0 - \omega}{2} t \right) \sin \left( \frac{\omega_0 + \omega}{2} t \right) \).

If \( \omega_0 \) is close to \( \omega \), then \( \frac{\omega_0 - \omega}{2} \) is small compared to \( (\omega_0 + \omega)/2 \). Then the graph will look like "beats."
The beat goes on ....

> plot(sin(20*t), t=0..10*Pi, numpoints=1000, title="y = sin(20t)");

> plot(sin(t), t=0..10*Pi, numpoints=1000, title="y = sin(t)";)

> plot(sin(t)*sin(20*t), t=0..10*Pi, numpoints=1000, title="y = sin(20t)sin(t)";)

y = sin(20t)

y = sin(t)

y = sin(20t)sin(t)
Case 2  \( w = w_0 \)

Let \( u_p = A t \cos(\omega_0 t) + B t \sin(\omega_0 t) \)

Plug into \( mu'' + ku_p = F_0 \cos(\omega_0 t) \) and you should get

\[
A = 0, \quad B = \frac{F_0}{2m\omega_0^2}.
\]

Thus, \( u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0^2} t \sin(\omega_0 t) \)

If we set \( u(0) = u'(0) = 0 \) then you should get \( C_1 = C_2 = 0 \).

The graph of \( u(t) = \frac{F_0}{2m\omega_0^2} t \sin(\omega_0 t) \) is

Resonance!
Forced Vibrations with Damping ($\delta > 0$).

We consider $m u'' + y u' + ku = F_0(t) = F_0 \cos(\omega t)$. \hspace{1cm} (*)

The solution of the homogeneous problem

$m u'' + y u' + ku = 0$ is

$$u_h = C_1 e^{\eta t} + C_2 e^{\bar{\nu} t} \begin{cases} \eta, & \text{both real and } < 0 \\ \eta \pm \sqrt{\eta^2 - 4\nu}, & \nu = -\frac{y^2}{2m} < 0 \end{cases}$$

$$C_1 e^{\eta t} + C_2 e^{\bar{\nu} t} \quad (\delta = -\frac{y^2}{2m} < 0),$$

or

$$\alpha e^{-\frac{\eta}{2} t} (C_1 \cos \beta t + C_2 \sin \beta t),$$

$$\beta = \sqrt{\frac{4\nu - \delta^2}{2m}}.$$

In all three cases $\lim_{t \to \infty} u_h(t) = 0$. Because of this $u_h$ is called the transient part of the solution.

The particular solution to (*) is of the form

$$u_p(t) = A \cos(\omega t) + B \sin(\omega t),$$

It is called the steady state part of the solution.

We will study how $u_p(t)$ changes with different values of $\omega$. 
We can rewrite \( y(t) = A \cos(\omega t) + B \sin(\omega t) \) in the form \( R \cos(\omega t - \delta) \). Later, we will derive these formulas

\[
R = \frac{Fo}{\Delta} \quad \cos \delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta} \quad \sin \delta = \frac{\omega}{\Delta}
\]

where \( \Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \) and \( \omega_0 = \sqrt{\frac{E}{m}} \).

This system no longer blows up when \( \omega_0 = \omega \) but \( R \) does hit its maximum when regarded as a function of \( \omega \) where \( \omega = \omega_0 \). One can check,

\[
\left. \frac{dR}{d\omega} \right|_{\omega_0} = 0 \quad \text{and} \quad R(\omega_0) = \frac{F_0}{\gamma \omega_0}
\]

(See graphs on page 211.)
Here is the derivation of the formulas (\( \text{E}\)).

\[
\begin{align*}
U_p &= A \cos \omega t + B \sin \omega t \\
U_p' &= -A \omega \sin \omega t + B \omega \cos \omega t \\
U_p'' &= -A \omega^2 \cos \omega t - B \omega^2 \sin \omega t
\end{align*}
\]

\[m U_p'' + y U_p' + k U_p = \]

\[
(-m A \omega^2 + y B \omega + k A) \cos \omega t + (m B \omega^2 - y A \omega + k B) \sin \omega t
\]

\[m u_s t = F_0 \cos (\omega t). \text{ Thus}
\]

\[\begin{align*}
(k - m \omega^2) A + y \omega B &= F_0 \\
\Rightarrow -y \omega A + (k - m \omega^2) B &= 0
\end{align*}\]

\[\begin{align*}
y u (k - m \omega^2) A + y^2 \omega^2 B &= y \omega F_0 \\
-y \omega (k - m \omega^2) A + (k - m \omega^2)^2 B &= 0
\end{align*}\]

\[
\left[ y^2 u^2 + (k - m \omega^2)^2 \right] B = y \omega F_0
\]

\[
B = \frac{y \omega F_0}{y^2 u^2 + (k - m \omega^2)^2}
\]

\[
A = \frac{(k - m \omega^2) B}{y \omega}
\]

Next we find \( R = \sqrt{A^2 + B^2} \).
\[ R^2 = \frac{(K - \omega^2 m)^2}{\gamma^2 \omega^2} \beta^2 + \beta^2 = \left[ \frac{(K - \omega^2 m)^2}{\gamma^2 \omega^2} + 1 \right] \beta^2 \]

\[ = \left[ \frac{(K - \omega^2 m)^2 + \gamma^2 \omega^2}{\gamma^2 \omega^2} \right] \frac{\gamma^2 \omega^2 \beta^2}{(\gamma^2 \omega^2 + (K - \omega^2 m^2)^2)} \]

\[ = \frac{\beta^2}{\gamma^2 \omega^2 + (K - \omega^2 m^2)^2} \]

Now, \((K - m \omega^2) = m\left(\frac{K}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2)\).

Let \[ \Lambda = \left(\gamma^2 \omega^2 + m^2(\omega_0^2 - \omega^2)^2\right)^{1/2} \]

Then \[ R^2 = \frac{\beta^2}{\Lambda^2} \Rightarrow R = \frac{\beta}{\Lambda}. \]

Now for \(d\),

\[ \sin d = \frac{\beta}{R} = \frac{\gamma \omega F_0}{\beta \Lambda} = \frac{\gamma \omega}{\Lambda} \]

\[ \cos d = \frac{A}{R} = \frac{m(\omega_0^2 - \omega^2) \beta}{\gamma \omega \beta} = \frac{F_0}{\beta \Lambda} \]

\[ = \frac{m(\omega_0^2 - \omega^2)}{\gamma \omega} \cdot \frac{\beta}{R} = \frac{m(\omega_0^2 - \omega^2)}{\Lambda}. \]
Example (3.8 #15) Find a differentiable solution to

\[ u'' + u = F(t), \quad u(0) = 0, \quad u'(0) = 0 \]

where

\[ F(t) = \begin{cases} 
F_0 \sin t & 0 \leq t < \pi \\
F_0 \sin(2\pi - t) & \pi \leq t < 2\pi \\
0 & 2\pi \leq t 
\end{cases} \]

Solution

First, I always like to graph the forcing function.

![Graph of F(t)]

We will divide the problem into three parts:

1. \( t \in [0, \pi] \),
2. \( t \in [\pi, 2\pi] \),
3. \( t \geq 2\pi \).

For 1, we find a solution using \( u(0) = u'(0) = 0 \). Call this solution \( u_1(t) \).

For 2, we find a solution, call it \( u_2(t) \), using

\[ u_2(\pi) = u_1(\pi), \quad u_2'(\pi) = u_1'(2\pi) \]

For 3, we find a solution, call it \( u_3(t) \), using

\[ u_3(2\pi) = u_2(2\pi), \quad u_3'(2\pi) = u_2'(2\pi) \]

Then our solution to the original problem is

\[ u(t) = \begin{cases} 
\frac{F_0}{2\pi} \sin t & t \in [0, \pi] \\
\frac{F_0}{2\pi} \sin(2\pi - t) & t \in [\pi, 2\pi] \\
0 & t \geq 2\pi 
\end{cases} \]

Got it?
Part 1 \( t \in \left(0, \pi\right) \) \[ u'' + u = F_o \quad u(0) = 0 \quad u'(0) = 0. \]

\[ u_h = C_1 \cos t + C_2 \sin t \quad u_p = F_o \cdot t. \]

The general solution is

\[ u_1 = C_1 \cos t + C_2 \sin t + F_o \cdot t. \]

\[ u_1(0) = C_1 \Rightarrow C_1 = 0 \]

\[ u_1'(t) = C_2 \cos t + F_o \]

\[ u_1'(0) = C_2 + F_o \Rightarrow C_2 = -F_o. \]

Thus

\[ u_1(t) = -F_o \sin t + F_o \cdot t. \]

Part 2 \( t \in \left(\pi, 2\pi\right) \)

\[ u_h = C_1 \cos t + C_2 \sin t \quad u_p = F_o \cdot (2\pi - t) \]

\[ u_2(t) = C_1 \cos t + C_2 \sin t + F_o \cdot (2\pi - t) \]

\[ u_2(\pi) = -C_1 + F_0 \pi \quad u_1(\pi) = F_0 \pi \]

Thus \( C_1 = 0 \).

\[ u_2'(t) = C_2 \cos t - F_0 \quad u_1'(t) = -F_0 \cos t + F_0 \]

\[ u_2'(0) = -C_2 - F_0 \quad u_1'(0) = F_0 + F_0 = 2F_0. \]

Thus \( C_2 = -3F_0 \).
Thus we get \( U_2(t) = -3F_0 \sin w t + F_0 (2\pi - t) \).

\[ U_3(t) = C_1 \cos t + C_2 \sin wt , \]

\[ U_3(2\pi) = C_1 \quad U_2(2\pi) = 0 \]

Thus \( C_1 = 0 \)

\[ U_3'(t) = C_2 \cos t \quad U_2'(t) = -3F_0 \cos t - F_0 \]

\[ U_3'(2\pi) = C_2 \quad U_2'(2\pi) = -4F_0 \]

Thus \( C_2 = -4F_0 \)

\[ U_3(t) = -4F_0 \sin wt \]

**Part 4**

We put it all together to get

\[ U(t) = \begin{cases} 
- F_0 \sin wt + F_0 t & t \in [0, \pi] \\
-3F_0 \sin wt + F_0 (2\pi - t) & t \in (\pi, 2\pi) \\
-4F_0 \sin wt & t \in (2\pi, 100) 
\end{cases} \]

See next page for the graphs.
Some plots for Example in class (3.8 #15)

> F := t -> piecewise(t>0 and t<Pi, t, t>:=Pi and t<2*Pi, 2*Pi-t); # use help
to lookup "piecewise"
> plot(F(t), t=0..10*Pi, color=black, thickness=3, title='Forcing Function');

Forcing Function

> u := t -> piecewise(t>0 and t<Pi, -sin(t) + t, t>:=Pi and
t<=2*Pi, -3*sin(t)+2*Pi-t, t>2*Pi, -4*sin(t));
> plot(u(t), t=0..10*Pi, color=black, thickness=3, title='Solution');

Solution

> evalf(2*Pi-solve(diff(-3*sin(t)+2*Pi-t, t)=0));
> evalf(u(%));
(dbm:7) \( F(t) := \) if \( 2^\pi \leq t \) then 0 else if \( \pi \leq t \) then \( 2^\pi - t \) else \( t \) $

(dbm:7) \text{wxplot2d}(F(t),[t,0.10^\pi])$

\[
\begin{align*}
\text{if } 2^\pi &\leq t \text{ then 0 else if } \pi \leq t \text{ then } 2^\pi - t \text{ else } t
\end{align*}
\]

(dbm:7) \( u(t) := \) if \( t \geq 2^\pi \) then \(-4^\sin(t)\) else if \( t \geq \pi \) then \(-3^\sin(t) - t + 2^\pi\) else \(-\sin(t) + t\) $

(dbm:7) \text{wxplot2d}(u(t),[t,0.10^\pi])$

\[
\begin{align*}
\text{if } 2^\pi &\geq t \text{ then } -4^\sin(t) \text{ else if } \pi \geq t \text{ then } -3^\sin(t) - t + 2^\pi \text{ else } -\sin(t) + t
\end{align*}
\]