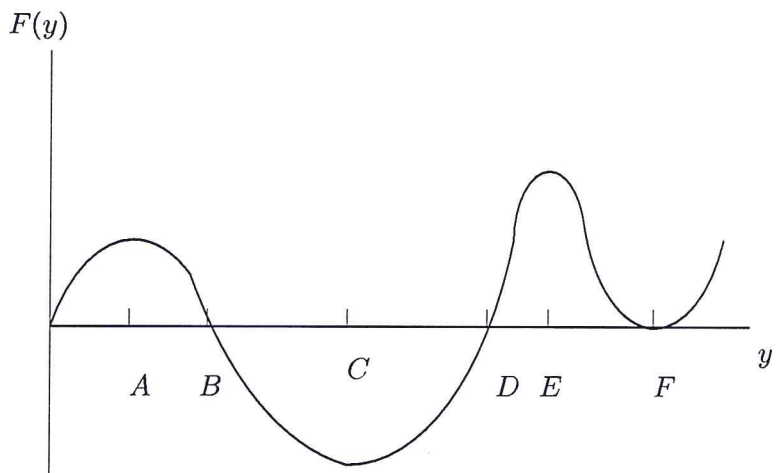
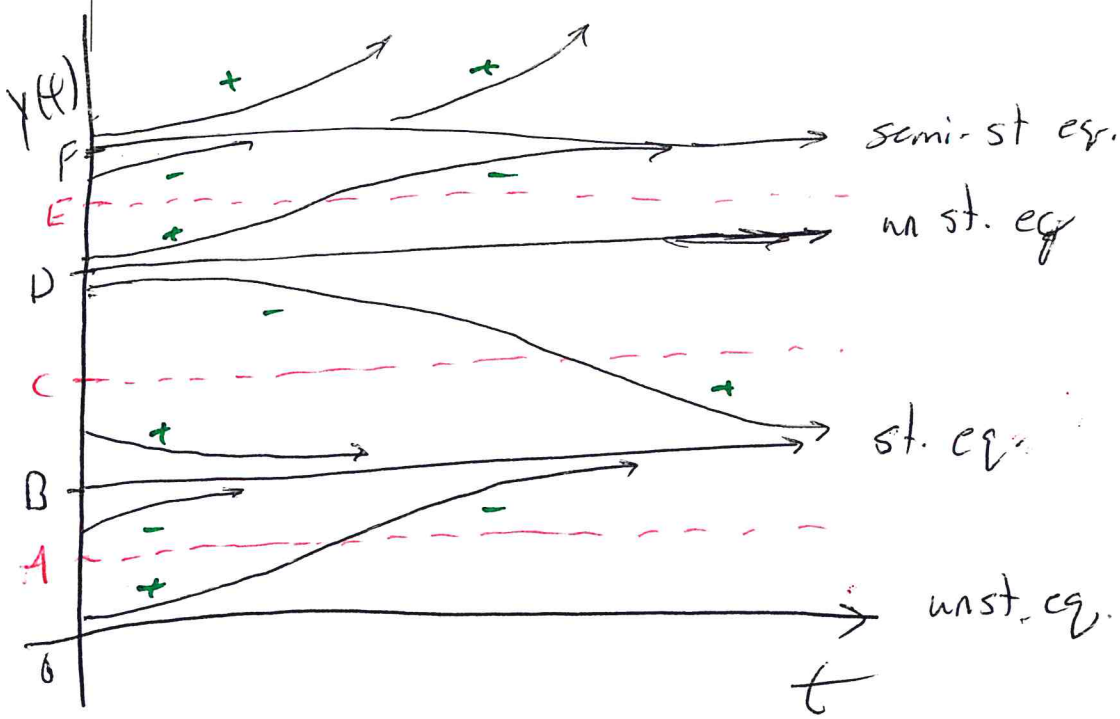
Math 305 Test 1 Spring 2017

Name: _____ Section time: _____

ONLY NON GRAPHING CALCULATORS ALLOWED

1. [10 points] Suppose $y'(t) = F(y(t))$, where the graph of $F(y)$ is given below. Carefully draw several solution curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the initial concavity of the solution curves. Assume $y(t)$ and t are non-negative.

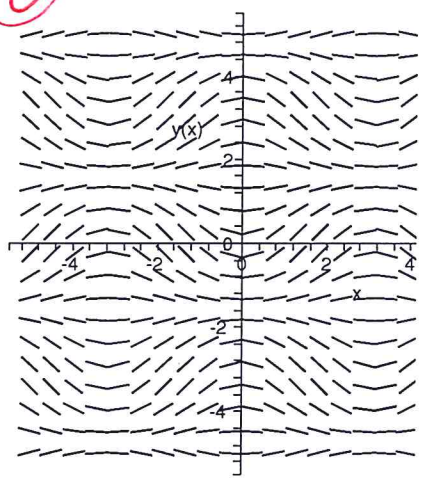




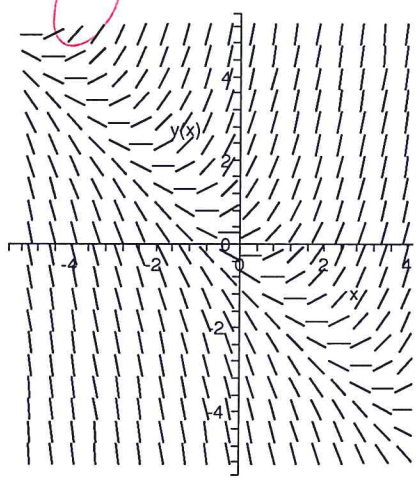
2. [10 points] You are in the middle of giving a presentation to a group of foreign investers in your new start up company when you realize that the Russians have hacked your laptop and deleted the labels on your direction field slides and inserted one slide your weren't going to use. Match the differential equations with their corresponding direction fields. (Each correct match is worth 2 points, each incorrect match is -1 point.)

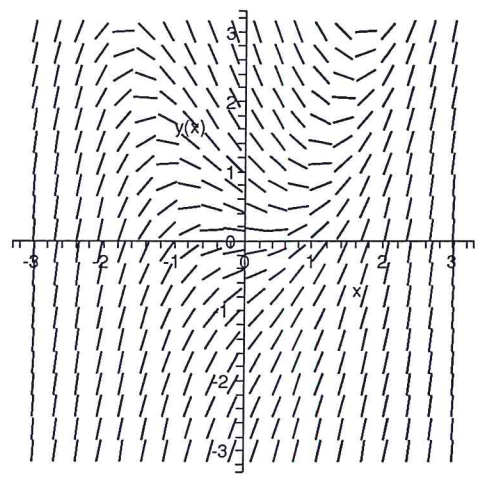
(a) $y' = x + y$
(b) $y' = y + 1$
(c) $y' = \sin x \cos y$
(d) $y' = y - 3$
(e) $y' = y - 2x$

c



a





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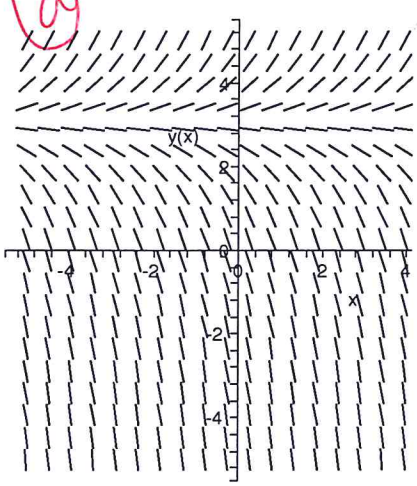

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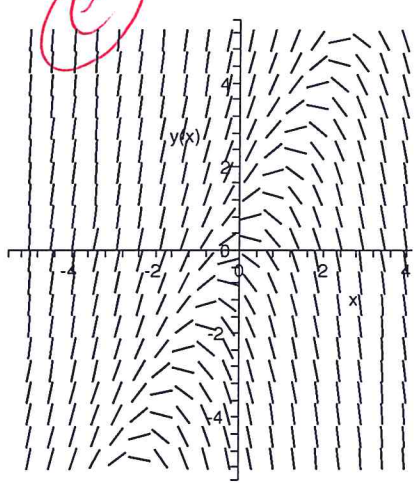
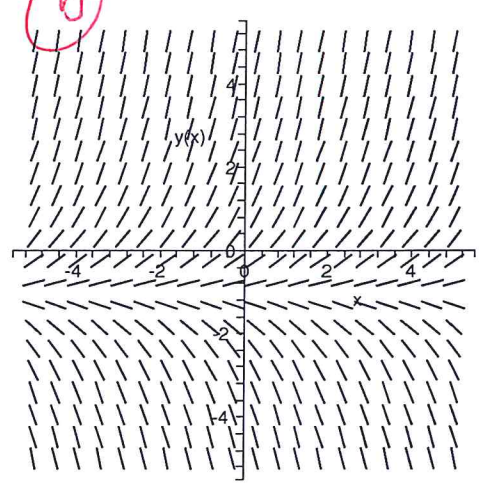


b



3. [10+2 points] a. Solve $\frac{dy}{dt} = y(K-y)$ where $K > 0$ and $y(0) = y_0 > 0$. Find y as a function of t. (The algebra is harder that the calculus here.)

$$\int \frac{1}{y(k-y)}\,dy = \int dt = t + C''$$

$$\int \frac{\frac{1}{k}}{y} + \frac{\frac{1}{k}}{k-y}\,dy = \frac{1}{k}\left(\ln|y| - \ln|k-y|\right) = k \ln\left|\frac{y}{k-y}\right|$$

Thus $\ln\left|\frac{y}{k-y}\right| = kt + C'$

$$\left|\frac{y}{k-y}\right| = e^{C'}e^{kt}$$

$$\frac{y}{k-y} = \pm e^{C}e^{kt} = Ce^{kt}$$

$y(0) = y_0 \Rightarrow C = \frac{y_0}{k-y_0}$

$$\frac{y}{k-y} = \frac{y_0}{k-y_0}e^{kt} = B$$

$$\frac{y}{k-y} = B$$

$$y = Bk - By$$

$$y + By = Bk$$

$$y = \frac{Bk}{1+B}$$

$$y = \frac{\frac{y_0 k}{k-y_0}e^{kt}}{1 + \frac{y_0}{k-y_0}e^{kt}}$$

$$= \boxed{\frac{y_0 k}{(k-y_0)e^{-kt} + y_0}}$$

b. What is $\lim_{t\to\infty} y(t)$?

lim as $t \to \infty$ is

$$\frac{y_0 k}{0 + y_0} = \boxed{k}$$

4. [10 points] Find the general solution to

$$\frac{dy}{dx} = \frac{y}{x} + \csc\left(\frac{y}{x}\right).$$

You may leave your answer as a relation between x and y.

Let $v = \frac{y}{x}$. Then $y = xv$ and $y' = v + xv'$.

Thus

$$\cancel{v} + xv' = \cancel{v} + \csc(v).$$

$$\int \sin(v)\, dv = \int \frac{1}{x}\, dx$$

$$-\cos(v) = \ln|x| + C,$$

$$\boxed{\cos\left(\frac{y}{x}\right) = \ln\left|\frac{1}{x}\right| + C.}$$

$\boxed{-\cos\left(\frac{y}{x}\right) = \ln|x| + C}$ is fine too.

5. [2+3+10 points] a. Show that

$$\underbrace{(3x^3y^2 + 2x^2y)}_{M} + \underbrace{(x^4y + 2xy^2)}_{N}y' = 0$$

is not exact.

$M_y = 6x^2y + 2x^2 \quad N_x = 4x^3y + 2y^2$. Not Exact

b. Multiply through by $1/xy$ and show that the new equation is exact.

$$\underbrace{(3x^2y + 2x)}_{M} + \underbrace{(x^3 + 2y)}_{N}y' = 0$$

$M_y = 3x^2 \qquad N_x = 3x^2$ exact.

c. Find the general solution. You may leave your answer as a relation.

$\psi = \int 3x^2y + 2x\,dx = x^3y + x^2 + C_1(y)$

$\psi = \int x^3 + 2y\,dy = x^3y + y^2 + C_2(x)$

Let $\psi = x^3y + x^2 + y^2$

Gen. Solution is $\boxed{x^3y + x^2 + y^2 = C}$

6. [10 points] Find the general solution to

$$y' + t^2 y = t^2 y^4.$$

Hint: It is a Bernoulli type equation.

Let $v = y^{1-n} = y^{1-4} = y^{-3}$.

Then $y = v^{-1/3}$, $y^4 = v^{-4/3}$ and $y' = -\frac{1}{3} v^{-\frac{4}{3}} v'$.

We now substitute to get

same!

$$-\frac{1}{3} v^{-\frac{4}{3}} v' + t^2 v^{-\frac{1}{3}} = t^2 v^{-\frac{4}{3}}$$

multiply through by $-3v^{4/3}$.

$$v' - 3t^2 v = -3t^2$$

This is linear.

$$\mu = e^{-3\int t^2 dt} = e^{-t^3}$$

$$(e^{-t^3} v)' = -3t^2 e^{-t^3}$$

$$e^{-t^3} v = \int -3t^2 e^{-t^3} dt$$

Let $u = -t^3$. Then $du = -3t^2$

$$= \int e^u du = e^u + C = e^{-t^3} + C$$

$$v = 1 + Ce^{t^3}$$

$$y = (1 + Ce^{t^3})^{-1/3}$$

(Notice, if $C = 0$, $y = 1$ is a solution.)

Notice $y = 0$ is a solution even though no value of C gives $y = 0$. We missed this case since we divided by y in our analysis.

7. [10+3 points] Find the general solution to $y'' + y' = x$. Then find the particular solution for $y(0) = 1$, $y'(0) = 2$. (The method we covered in Chapter 3 won't work because this equation is nonhomogeneous.)

Let $w = y'$. Then we have $w' + w = x$, which is linear

$\mu = e^{\int 1\,dx} = e^x$. $\quad e^x w' + e^x w = xe^x$

$$(e^x w)' = xe^x$$

$$e^x w = \int xe^x\,dx = (x-1)e^x + C_1$$

$$w = -1 + x + C_1 e^{-x}$$

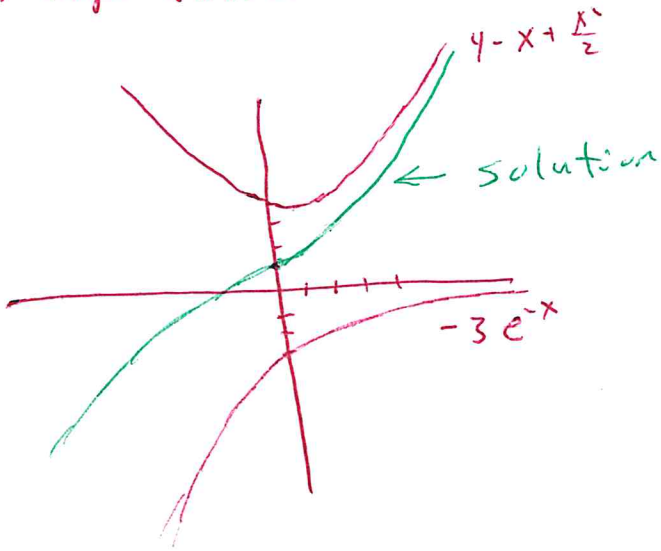
$$y = \int w\,dx = \int -1 + x + C_1 e^{-x}\,dx = -x + \frac{x^2}{2} - C_1 e^{-x} + C_2$$

$$y(0) = 1 \Rightarrow -C_1 + C_2 = 1$$

$$2 = y'(0) = w(0) = -1 + C_1 = 2 \Rightarrow C_1 = 3 \Rightarrow C_2 = 4$$

Thus $\boxed{y = 4 - x + \frac{x^2}{2} - 3e^{-x}.}$

Note: For large values of $x \gg 0$ we could use $y(x) \approx 4 - x + \frac{x^2}{2}$.



$Q(0) = 0$

8. [15 points] A pond has 100 gal of freshwater. At $t = 0$ brackish water starts to spill in at 2 gal/hour. Its salinity is α lb/gal. The well mixed water flows out of tank at the same rate. After fifty minutes the salinity of the tank water is measured and found to be 0.25 lb/gal. Find α. (Watch your algebra!)

$\frac{dQ}{dt}$ = rate in − rate out

$2\alpha - 2\frac{Q}{V}$ $\qquad V = 100$ fixed.

$Q' = 2\alpha - \frac{Q}{50}$

$Q' + \frac{Q}{50} = 2\alpha$ Linear. (It is also separable.)

$\mu = e^{\frac{t}{50}}$

$e^{t/50} Q' + e^{t/50} \frac{Q}{50} = 2\alpha e^{t/50}$

$(e^{t/50} Q)' = 2\alpha e^{t/50}$

$e^{t/50} Q = \int 2\alpha e^{t/50} dt = 100\alpha e^{t/50} + C$

$Q = 100\alpha + Ce^{-t/50}$ $\qquad Q(0) = 0 \Rightarrow C = -100\alpha,$

$Q = 100\alpha(1 - e^{-t/50})$

50 min = $\frac{5}{6}$ hour $\qquad Q(\frac{5}{6}) = \frac{1}{4} \times V = 25$

$100\alpha(1 - e^{-\frac{1}{60}}) = 25$

$\boxed{\alpha = \frac{1}{4}(1 - e^{-\frac{1}{60}})^{-1} = 15.12534722\ldots}$

9. [10 BONUS points] In this problem you are to find the general solution to

$$\frac{dy}{dx} = \frac{x}{x^2y + y^3} \qquad (*)$$

in the form of a relation between x and y. None of our methods will work. Instead let $u = x^2$ and the use fact that $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ to convert $(*)$ into an equation involving only u and y. Then use the fact that

$$\frac{du}{dy} = \frac{1}{\frac{dy}{du}}$$

to rewrite the equation with u a function of y. It will be linear in u' and u. Find its general solution. Then convert u back to x^2.

You may use $\int y^3 e^{-y^2}\, dy = \frac{-1}{2}(1+y^2)e^{-y^2} + C$.

$u = x^2 \qquad x = \sqrt{u}$

$\frac{dy}{dx} = \frac{\sqrt{u}}{uy + y^3} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du} 2\sqrt{u}$

$2\cancel{\sqrt{u}} \frac{dy}{du} = \frac{\cancel{\sqrt{u}}}{u^2 y + y^3}$

$\frac{1}{2}\frac{du}{dy} = u^2 y + 2y^3$

$u' - 2uy = 2y^3 \qquad a = e^{-\int 2y\,dy} = e^{-y^2}$

$(e^{-y^2} u)' = 2y^3 e^{-y^2}$

$e^{-y^2} u = -\frac{1}{2}(1+y^2)e^{-y^2} + C \qquad y(0) = 1$

$u = -(1+y^2) + Ce^{y^2}$

$x^2 = -(1+y^2) + Ce^{y^2}$

$0 = -2 + C$