1. [15 points] Find the general solution to $y'' + 3y' - 4y = \sin x$.

\[ r^2 + 3r - 4 = 0 \]
\[ (r + 4)(r - 1) = 0 \]
\[ r = -4, 1 \]

Let $y_h = C_1 e^{-4x} + C_2 e^x$.

Then $y_p = A \sin x + B \cos x$.

Let $y_p' = A \cos x - B \sin x$ and $y_p'' = -A \sin x - B \cos x$.

Plug $y_p$ in given ee. ($f = \sin x$, $c = \cos x$)

\[-A \sin x - 3A \cos x - 3B \sin x - 4A \cos x - 4B \sin x = 5 \sin x\]

\[\left(-A - 3B - 4A\right) + \left(-3A + 3A - 4B\right) c = 5 \sin x\]

\[\begin{align*}
-5A - 3B &= 5 \\
3A - 5B &= 0
\end{align*}\]

\[\begin{align*}
-15A - 9B &= 3 \\
15A - 25B &= 0
\end{align*}\]

\[\begin{align*}
-34B &= 3 \\
B &= \frac{-3}{34}
\end{align*}\]

\[A = \frac{-5}{34}\]

General solution is:

\[y = C_1 e^{-4x} + C_2 e^x - \frac{5}{34} \sin x + \frac{3}{34} \cos x\]

2. [10 points] Find the general solution to $y''' - y'' + 2y' - 2y = 0$.

\[r^3 - r^2 + 2r - 2 = 0\]
\[r^2(1-r) + 2(r-1) = 0\]
\[(r^2+2)(r-1) = 0\]

\[r = \pm \sqrt{2}, r = 1\]

\[y = C_1 e^x + C_2 \sin(\sqrt{2}x) + C_3 (\sqrt{2} x)\]
3. [15 points] Find the general solution to $y'' - 4y' + 4y = e^{2x}$

\[ r^2 - 4r + 4 = 0 \quad \text{Sol. of } y'' + 4y' + 4y = 0 \]

\[ (r - 2)^2 = 0 \quad \text{is } C_1 e^{2x} + C_2 xe^{2x} = y_h. \]

\[ r = 2, 2 \]

For $y_p$, if you use $Ae^{2x}$ it cannot work since $e^{2x}$ is a solution of homogeneous case. Likewise $Ax e^{2x}$ will not work. You have use $y_p = Ax^2 e^{2x}$.

Now plug and find $A$. $y_p' = 2Ax e^{2x} + 2Ax^2 e^{2x}$

\[ y_p'' = 2A e^{2x} + 8Ax e^{2x} + 4Ax^2 e^{2x}. \]

\[ 2A e^{2x} + 8Ax e^{2x} + 4Ax^2 e^{2x} = 2A e^{2x} + 8Ax e^{2x} - 8Ax e^{2x} - 8Ax e^{2x} + 4Ax^2 e^{2x} = e^{2x} \]

Notice only one term is a number times $e^{2x}$. Hence $A = \frac{1}{8}$.

The other terms will all cancel out.

\[ y = C_1 e^{2x} + C_2 xe^{2x} + \frac{1}{8} x^2 e^{2x} \]

4. [5 points] Suppose we given $y'' + p(t)y' + q(t)y = 0$, with $p(t)$ and $q(t)$ continuous everywhere, and are told that $t(t + 1)(t - 1)$ and $t^2 + 1$ are solutions. Is this possible? Explain. Hint: graph them.

They cannot both be solutions since they do not interlace. See Lecture Notes for 3.2, Theorem 9.
5. [10 points] Find the first five terms of the Taylor polynomial of the solution to

\[ y'' + xy' - 2y = 0, \quad y(0) = 1, y'(0) = 0. \]

\[
\begin{align*}
\overline{a_0} &= 1, \quad \overline{a_1} = 0, \\
\overline{y'} &= 2y - xy' \\
\overline{y''(0)} &= 2y(0) - 0 \cdot y'(0) = 2, \quad a_2 = \frac{y''(0)}{2!} = \frac{2}{2} = 1, \\
\overline{y'''(0)} &= 2y' - y' - xy'' \\
&= y' - xy'' \\
&= \overline{y''} \\
\overline{y''''(0)} &= \overline{y'''} - 0 \cdot \overline{y''}(0) = 0 - 0 \cdot 2 = 0, \quad a_3 = \frac{y''''(0)}{3!} = 0, \\
\overline{y'''''}(0) &= \overline{y''''} - xy''' \\
&= xy'' \\
\overline{y''''''(0)} &= 0, \quad a_4 = \frac{y''''''(0)}{4!} = 0. \\
\end{align*}
\]

In fact, all \( a_n, \ n > 4, \) are also zero.

The solution is \( y = 1 + x^2. \)
6. [5 points] Convert $3 \cos 2t + 4 \sin 2t$ into the form $R \cos (w_0 t - \delta)$. (Find \( \delta \) in radians.)

\[
R = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.
\]

\[
S = \tan^{-1} \left( \frac{4}{3} \right) = 0.927952 - \text{rad.}
\]

("No need to add \( \pi \)."
\[w_0 = 2.\]
\[5 \cos \left( 2t - 0.927952 \right) \]

7. [5 points] Find the general solution to $y'' + y = 0$. (This is a freebie.)

\[y = C_1 \cos x + C_2 \sin x\]

8. [5 points] Suppose we are given $y'' + y = t(1 + \sin t)$. To apply the Method of Undetermined Coefficients what should the form of the particular solution be?

\[y'' + y = t + ts \sin t.\]

Let $Y_p = At + B$. Now $Y_p = (E + D) \sin t + (E + F) \cos t$

It turns out the general solution is
\[y = C_1 \cos t + C_2 \sin t + \frac{1}{4} t \sin t - \frac{1}{4} t^2 \cos t\]

\[Y_p = At + B + (C_1 + D) \sin t + (C_2 + E) \cos t.\]

Not required

Instead use
\[Y_p = (C_1 + D) \sin t + (C_2 + E) \cos t.\]
This #8 done on Maple. Notice, it has a redundant term, $\frac{1}{4} \cos(t)$. One should rewrite this as

$$y(t) = C_1 \cos(t) + C_2 \sin(t) + \frac{1}{4} t \sin(t) - \frac{1}{4} t^2 \cos(t).$$
9. [15 points] Show that \( y_1 = x \) is a solution of
\[
x^2 y'' - x(x + 2)y' + (x + 2)y = 0.
\]

To find a second linearly independent solution let \( y_2 = v(x)y_1 = vx \).
Substitute this into the differential equation to get a differential equation in \( v \). Solve it. This is the \textit{Reduction of Order Method}.

Let \( y = vx \). Then \( y' = v + vx' \) and \( y'' = v' + v'x + v''x \).

Plug these into the given equation:
\[
x^2(2v' + xv'') - x(x+2)(v + xv') + (x+2)xv'' = 0
\]

\[
\Rightarrow \quad x^3 v'' + \left[2x^2 - x^3 - 2x^2\right] v' + \left[-\frac{x(x+2) + x^2}{x^3}\right] v = 0
\]

\[
\Rightarrow \quad x^3 v'' - x^3 v' = 0
\]

\[
\Rightarrow \quad v'' - v' = 0
\]

Let \( w = v' \). Then
\[
w' - w = 0 \quad \Rightarrow \quad w = w.
\]
Thus \( w = C_1 e^x \). Use \( C_1 \), \( C_1 = 1 \).

\[
v = \int w \, dx = e^x + C_2 \quad \Rightarrow \quad \text{Use } e^x, C_2 = 0.
\]

Then \( y_2 = xv = xe^x \).

It is clear that \( \{x, xe^x\} \) is linearly independent.

Thus the general solution is
\[
y = C_1 x + C_2 xe^x.
\]
10. [15 points] A 4 kilogram object is attached to the lower end of a spring whose upper end is attached to the ceiling. The spring constant \( k \) is 2 kg/meter. The resistance to the motion is \( \gamma = 2 \text{ N-s/m} \) times the velocity of the object. The object is set in motion by pushing it up 0.5 m and then letting it go.

Set up and solve a differential equation to model this mass-spring system. Is this system an example of small-damping, over-damping or critical-damping?

\[
4u'' + 2u' + 2u = 0 \quad u(0) = -\frac{1}{2} \quad u'(0) = 0.
\]

\[
2u'' + u' + u = 0
\]

\[
2r^2 + r + 1 = 0
\]

\[
r = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot 1}}{4} = -\frac{1 \pm i\sqrt{7}}{4}
\]

General solution is \( u(t) = e^{\frac{-t}{4}} (C_1 \cos \frac{\sqrt{7}}{4} t + C_2 \sin \frac{\sqrt{7}}{4} t) \).

Since \(-\frac{1}{4} < 0\) it is small-damping.

\[
u(0) = C_1 = -\frac{1}{2}.
\]

\[
u'(t) = -\frac{1}{4} e^{\frac{-t}{4}} (C_1 \cos \frac{\sqrt{7}}{4} t + C_2 \sin \frac{\sqrt{7}}{4} t) + C_2 e^{\frac{-t}{4}} \left( -\frac{\sqrt{7}}{4} \sin \frac{\sqrt{7}}{4} t \right)
\]

\[
u'(0) = -\frac{1}{4} (C_1 + 0) + 1 \left( C_2 \frac{\sqrt{7}}{4} \right) = 0
\]

\[
C_2 \frac{\sqrt{7}}{4} = -\frac{1}{8}
\]

\[
C_2 = -\frac{1}{2} \cdot \frac{1}{8} = -\frac{17}{16}
\]

\[
u(t) = e^{\frac{-t}{4}} \left( -\frac{1}{2} \cos \frac{\sqrt{7}}{4} t - \frac{17}{16} \sin \frac{\sqrt{7}}{4} t \right)
\]

\[
= \Gamma \frac{1}{2} e^{\frac{-t}{4}} \cos (\frac{\sqrt{7}}{4} t - 3.502655726\ldots)
\]

\( \mod{\text{not required.}} \)