Math 305 Test 3 Spring 2017

Name: ______________________ Section time: ______________________

Only Scientific Calculators are Allowed

1. [15 points] Suppose $y'' - \frac{1}{t-3}y' - 2ty = 0$ and that $y(0) = 2$ and $y'(0) = 1$.

   a. Find the first five terms of the power series solution for $y(t)$ centered about $t = 0$.

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots \qquad a_n = \frac{y^{(n)}(0)}{n!}$$

$$y(0) = 2 \Rightarrow a_0 = 2 \qquad y'(0) = 1 \Rightarrow a_1 = 1.$$

$$y'' = \frac{1}{t-3}y' + 2ty \qquad y''(0) = \frac{-1}{3}\cdot 1 + 2\cdot 0\cdot 2 = \frac{-1}{3} \qquad a_2 = \frac{-\frac{1}{3}}{2!} = \frac{-1}{6}$$

$$y''' = \frac{-1}{(t-3)^2}y' + \frac{1}{t-3}y'' + 2y + 2ty'$$

$$y'''(0) = \frac{-1}{9}\cdot 1 - \frac{1}{3}\left(\frac{-1}{3}\right) + 2\cdot 2 + 2\cdot 0\cdot 1 = 4 \qquad a_3 = \frac{4}{3!} = \frac{2}{3}$$

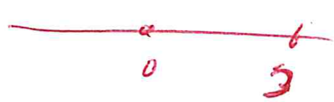
$$y'''' = \frac{2}{(t-3)^3}y' + \frac{-1}{(t-3)^2}y'' + \frac{-1}{(t-3)^2}y'' + \frac{1}{t-3}y''' + 2y' + 2y' + 2ty''$$

$$y''''(0) = \frac{2}{27}\cdot 1 - \frac{2}{9}\left(\frac{-1}{3}\right) + \frac{-1}{3}\cdot(4) + 4\cdot 1 + 2\cdot 0\cdot\left(\frac{-1}{3}\right)$$

$$= -\frac{4}{3} + \frac{4\cdot 3}{3} = \frac{8}{3} \qquad a_4 = \frac{8/3}{4!} = \frac{1}{9}$$

$$y \approx 2 + t - \frac{t^2}{3} + \frac{2t^3}{3} + \frac{t^4}{9}$$

   b. Give a lower bound for the radius of convergence.



$R \geq 3$

$\frac{1}{t-3}$ has a singularity at $t=3$. Distance to 0 is 3.

Thus $R \geq 3$.

2. [15 points] Consider $y'' + xy' - 2y = 0$.

a. Find the general power series solution centered about zero. Include a recursive formula for a_n.

Let $y = \sum_{n=0}^{\infty} a_n x^n$ $\quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$ $\quad xy' = \sum_{n=0}^{\infty} n a_n x^n$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n .$$

$$y'' + xy' - 2y = \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n - 2 a_n \right] x^n = 0$$

$n \geq 0$

Thus, $a_{n+2} = \dfrac{(2-n) a_n}{(n+2)(n+1)}$. or $a_n = \dfrac{(4-n) a_{n-2}}{n(n-1)}$, $n \geq 2$.

b. Now suppose $y(0) = 1$ and $y'(0) = 0$. Find the values of all the a_n's. (The series should terminate quickly.)

$a_0 = 1$ $\qquad a_1 = 0$

$a_2 = \frac{2}{2} = 1$ $\qquad a_3 = \frac{1 \cdot 0}{3 \cdot 2} = 0$

$a_4 = \frac{(4-4) a_2}{4 \cdot 3} = 0$ $\qquad a_5 = \frac{-1 \cdot 0}{4 \cdot 3} = 0$

$a_6 = \frac{0}{\ } = 0$ $\qquad a_7 = \frac{0}{\ } = 0$

$a_8 = \frac{(\ ) \cdot 0}{(\ )(\ )} = 0$

all of the odd term are zero.

all the rest are zero.

$y = x^2 + 1$ is the solution.

3. [10 points] Consider the partial differential equation below.

$$u_{xxt} - u_{xt} = u_x,$$

where u is a function of independent variables x and t. Suppose $u(x,t) = X(x)T(t)$. Derive the ordinary differential equations below.

$$X'' + (1 - a)X' = 0 \ \& \ aT' - T = 0.$$

typo.

$$u_{xxt} = X''T' \quad u_{xt} = X'T' \quad u_x = X'T$$

$$X''T' - X'T' = X'T$$

$$a = \frac{X'' - X'}{X'} = \frac{T}{T_1} = a$$

$$aX' = X'' - X'$$

$$X'' - X' - aX' = 0$$

$$X'' - (1+a)X' = 0.$$

$$T = aT_1$$

$$0 = aT_1 - T.$$

4. [10 points] Find all positive values of γ such that the boundary value problem

$$y''(t) + \gamma^2 y(t) = 0, \ \ y(0) = y(2) = 0,$$

has nontrivial solutions.

$\gamma^2 > 0 \Rightarrow y = C_1 \sin \gamma t + C_2 \cos \gamma t.$

$y(0) = 0 \quad y(0) = C_2 \Rightarrow C_2 = 0$

$y(2) = 0 \quad y(2) = C_1 \sin(\gamma 2)$

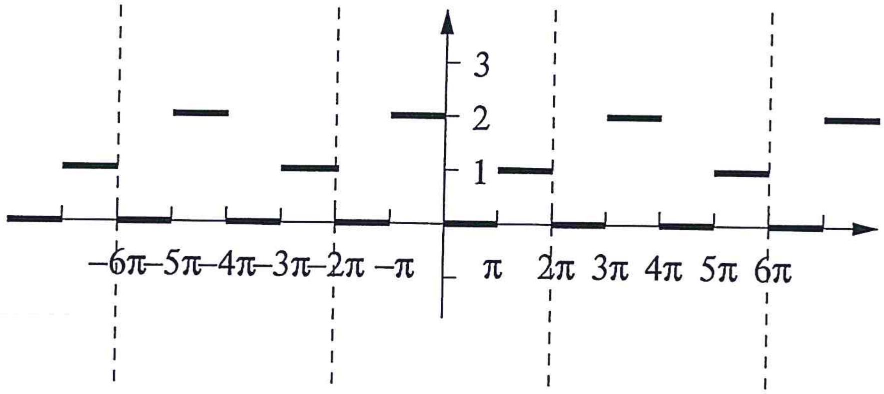
$C_1 \sin(2\gamma) = 0$ if and only if $C_1 = 0$ (trivial)

or $2\gamma = n\pi \quad n = 1, 2, 3 \ldots$

($\gamma > 0$ is given)

Thus, $\gamma = \frac{n\pi}{2}, \ n = 1, 2, 3 \ldots$

5. [20 points] Consider the periodic function $f(x)$ in the graph below.



We wish to calculate a few terms of its Fourier series:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right).$$

o. What is L? $\boxed{2\pi}$

i. What is a_0? (Be careful.)

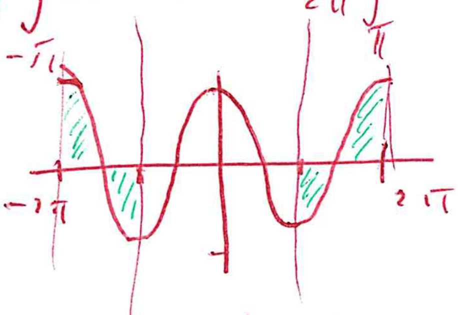
ii. What is a_2?

iii. What is b_2?

$\frac{a_0}{2}$ = ave value = $\frac{0+2+0+1}{4} = \frac{3}{4}$

Thus $\boxed{a_0 = 3/2.}$

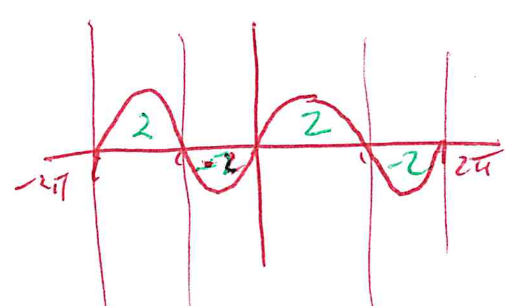
$$a_2 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\cos\left(\frac{2\cancel{\pi}x}{2\cancel{\pi}}\right)dx = \frac{1}{2\pi}\int_{-\pi}^{0} 2\cos x\,dx + \frac{1}{2\pi}\int_{\pi}^{2\pi} 1\cdot\cos x\,dx$$



Thus, both integrals are zero.

$\boxed{a_2 = 0.}$

$$b_2 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(x)\sin\left(\frac{2\cancel{\pi}x}{2\cancel{\pi}}\right)dx = \frac{1}{2\pi}\int_{-\pi}^{0} 2\sin x\,dx + \frac{1}{2\pi}\int_{\pi}^{2\pi} 1\cdot\sin x\,dx$$



$b_2 = \frac{1}{2\pi}2\cdot(-2) + \frac{1}{2\pi}1\cdot(-2)$

$\boxed{b_2 = -\frac{2}{\pi} + \frac{-1}{\pi} = \frac{-3}{\pi}}$

6. [4 points] Suppose $f(x)$ is an even function and $g(x)$ is an odd function. Prove that $h(x) = f(f(x)) + g(x)g(x)$ is even.

$$h(-x) = f(f(-x)) + g(-x)g(-x)$$
$$= f(f(x)) + (-1)g(x)(-1)g(x)$$
$$= f(f(x)) + g(x)g(x) = h(x).$$

7. [6 points] Let $f(x)$ be any function. Define

$$g(x) = \frac{f(x) - f(-x)}{2} \quad \& \quad h(x) = \frac{f(x) + f(-x)}{2}.$$

Show the following.

a. $g(x) + h(x) = f(x)$.

b. $g(x)$ is odd.

c. $h(x)$ is even.

Thus, g and h are called, respectively, the odd and even parts of f.

$$g + h = \frac{f(x) - f(-x) + f(x) + f(-x)}{2} = \frac{2f(x)}{2} = f(x)$$

$$g(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -g(x)$$

$$h(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = h(x).$$

8. [10 points] Find the solution to the heat conduction problem

$$u_{xx} = 16u_t, \quad 0 < x < 1, \quad t > 0;$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0;$$

$$u(x,0) = \sin(3\pi x) + 19\sin(6454\pi x).$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{1}\right) e^{-\frac{n^2\pi^2}{4^2}t}$$

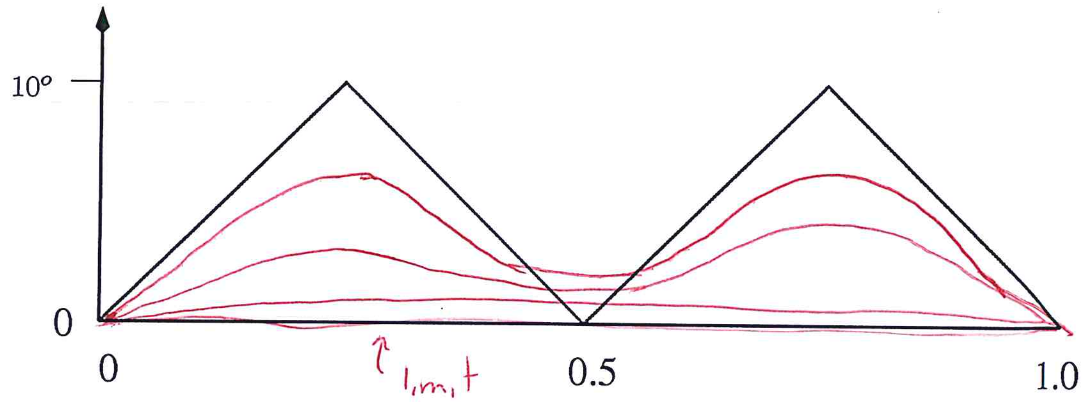
$c_n = 0$ except for c_3 and c_{6454}

$c_3 = 1 \quad c_{6454} = 19.$

$$\sin(\pi x) e^{-\frac{\pi^2}{16}t} + 19\sin(6454\pi x) e^{-\frac{(6454)^2\pi^2}{16}t}$$

9. [10 points]

a. A one meter metal rod initially has the temperature distribution shown below. Suppose the ends are held at 0 degrees. As $t \to \infty$ what will the temperature distribution approach? Graph this and sketch graphs for several intermediate temperature distributions we would expect to see.



b. A one meter metal rod initially has the temperature distribution shown below. Suppose the ends are insulated. As $t \to \infty$ what will the temperature distribution approach? Graph this and sketch graphs for several intermediate temperature distributions we would expect to see.