Math 305 Test 2 Spring 2018

Name: ______________________________

**ONLY NON GRAPHING CALCULATORS ALLOWED**

1. [15 points] Find the general solution to each of the following.

   a. $y'' + 2y' = 0$.

$r^2 + 2r + 0 = 0$

$r(r+2) = 0$

$r = 0, -2$

$$\boxed{y = C_1 + C_2 e^{-2t}}$$

   b. $y''' - y'' - 5y' - 3y = 0$. Hint: $r = 3$ is a root.

$$r - 3 \,\overline{\big)\, r^3 - r^2 - 5r - 3}\quad \text{quotient: } r^2 + 2r + 1 \longrightarrow = (r+1)^2$$

roots are $3, -1, -1$.

$$\boxed{y = C_1 e^{-t} + C_2 t e^{-t} + C_3 e^{3t}}$$

   c. $y'' + 2y' + 5y = 0$.

$r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4\cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$\boxed{y = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)}$$

2. [10 points] Find the general solution to $y'' + 2y' + y = e^{2t}$.

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1.$ $\quad y_h = C_1 e^{-t} + C_2 t e^{-t}.$

$y_p = A e^{2t}$

$y_p' = 2A e^{2t}$

$y_p'' = 4A e^{2t}$

$y_p'' + 2y_p' + y_p = (4A + 2 \cdot 2A + A) e^{2t}$

$9A e^{2t}.$

We need $9A e^{2t} = e^{2t}.$

Thus, $A = \frac{1}{9}.$

$$\boxed{y = y_h + y_p = C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{9} e^{2t}}$$

3. [15 points] Find the general solution to $y'' + y = \sin t + \cos t$.

You should know by now that $Y_h = C_1 \cos t + C_2 \sin t$.

The steps are: $r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

Hence $Y_h = C_1 e^{0t} \cos t + C_2 e^{0t} \sin t$

$= C_1 \cos t + C_2 \sin t$.

For $Y_p$ use

$Y_p = At \sin t + Bt \cos t$.

$Y_p' = A \sin t + At \cos t + B \cos t - Bt \sin t$

$Y_p'' = A \cos t + A \cos t - At \sin t - B \sin t - B \sin t - Bt \cos t$

$Y_p'' + Y_p = 2A \cos t - 2B \sin t$ must $=$ $\sin t + \cos t$

Thus $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

$\boxed{Y = C_1 \cos t + C_2 \sin t + \frac{1}{2} t \sin t - \frac{1}{2} t \cos t}$

4. [20 points] Solve each of the initial value problems.

a. $y'' - y' - 2y = 0$, $y(0) = y'(0) = 1$.

$r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$
$r = 2, -1$

$y = C_1 e^{-t} + C_2 e^{2t}$

$y(0) = C_1 + C_2 = 1$
$y'(t) = -C_1 e^{-t} + 2C_2 e^{2t}$
$y'(0) = -C_1 + 2C_2 = 1$

add these

$3C_2 = 2$
$C_2 = \frac{2}{3}$,
$C_1 + \frac{2}{3} = 1$.
so $C_1 = \frac{1}{3}$.

$$\boxed{y = \frac{1}{3} e^{-t} + \frac{2}{3} e^{2t}}$$

b. $y'' - y' - 2y = t$, $y(0) = y'(0) = 1$.

From a, $y_h = C_1 e^{-t} + C_2 e^{2t}$

Let $y_p = At + B$.
Then $y_p' = A$
and $y_p'' = 0$

$y_p'' - y_p' - 2y_p = 0 - A - 2At - 2B \overset{\text{must}}{=} t$
$-2At = t$
$A = -\frac{1}{2}$.
$-(-\frac{1}{2}) - 2B = 0$
$2B = \frac{1}{2}$
$B = \frac{1}{4}$

General sol. is
$y = y_h + y_p = C_1 e^{-t} + C_2 e^{2t} - \frac{1}{2}t + \frac{1}{4}$.

Now find $C_1$ & $C_2$. $y(0) = C_1 + C_2 + \frac{1}{4} = 1$, $C_1 + C_2 = \frac{3}{4}$
$y'(t) = -C_1 e^{-t} + 2C_2 e^{2t} - \frac{1}{2}$
$y'(0) = -C_1 + 2C_2 - \frac{1}{2} = 1$ $\quad -C_1 + 2C_2 = \frac{3}{2}$

add these

$3C_2 = \frac{3}{4} + \frac{3}{2}$
$C_2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$.
Thus $C_1 + \frac{3}{4} = \frac{3}{4} \Rightarrow C_1 = 0$

$$\boxed{y = \frac{3}{4} e^{2t} - \frac{1}{2}t + \frac{1}{4}}$$

5. [10 points] A mass-spring system has mass $m = 9$, damping $\gamma = 36$, and spring constant $k = 37$. (Don't worry about the units.) There is no external forcing.

a. Write down the differential equation for this system.

$$mu'' + \gamma u' + ku = 0$$

$$9u'' + 36u' + 37u = 0$$

b. What is the general solution?

$$9r^2 + 36r + 37 = 0$$

$$r = \frac{-36 \pm \sqrt{36^2 - 4 \cdot 9 \cdot 37}}{18} = \frac{-36 \pm 6\sqrt{36-37}}{18}$$

$$= -2 \pm \frac{1}{3} i.$$

$$\boxed{\text{Thus, } u(t) = C_1 e^{-2t} \cos\left(\tfrac{1}{3}t\right) + C_2 e^{-2t} \sin\left(\tfrac{1}{3}t\right)}$$

c. Is this system under damped, over damped or critically damped?

You will see oscillations.

6. [10 points] Consider $y'' - x^2y' + e^x y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 2$. Assume that the solution has a Taylor series:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{y^{[n]}(0)}{n!} \cdot x^n.$$

$a_0 = \frac{y(0)}{0!} = \frac{1}{1} = 1$

$a_1 = \frac{y'(0)}{1!} = \frac{2}{1} = 2$

Find the first five terms: $a_0$, $a_1$, $a_2$, $a_3$, $a_4$.

$a_2 = \frac{y''(0)}{2!}$.

$y'' = x^2 y' - e^x y$

$y''(0) = 0 - 1 \cdot y(0) = -1$. $\quad a_2 = \frac{-1}{2!} = -\frac{1}{2}$

$y''' = (x^2y' - e^x y)' = 2xy' + x^2y'' - e^x y - e^x y'$

$y'''(0) = 0 + 0 - 1 \cdot y(0) - 1 \cdot y'(0) = -1 - 2 = -3$.

$a_3 = \frac{y'''(0)}{3!} = \frac{-3}{3!} = -\frac{1}{2}$

$y'''' = (2xy' + x^2y'' - e^x y - e^x y')'$

$= 2y' + 2xy'' + 2xy'' + x^2y''' - e^x y - e^x y' - e^x y' - e^x y''$

$y''''(0) = 2y'(0) + 0 + 0 + 0 - 1 \cdot y(0) - 2y'(0) - y''(0)$

$= 4 \quad -1 \quad -4 \quad -(-1) = 0$

$a_4 = \frac{y''''(0)}{4!} = \frac{0}{4!} = 0$.

$$\boxed{y(x) \approx 1 + 2x - \frac{1}{2}x^2 - \frac{1}{2}x^3}$$