Methods for First Order Differential Equations

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1. Separable. $$\frac{dy}{dx} = g(y)h(x)$$

$$\int \frac{1}{g(y)}\,dy = \int h(x)\,dx$$

Then integrate both sides and solve for y if possible.

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2. Homogeneous. $$\frac{dy}{dx} = f\left(\tfrac{y}{x}\right)$$

Let $v = y/x$. Then $y = xv$ and $y' = v + xv'$. Substitution then gives the following.

$$v + xv' = f(v)$$

$$x\frac{dv}{dx} = f(v) - v$$

Which is separable!

$$\int \frac{1}{f(v) - v}\,dv = \int \frac{1}{x}\,dx = \ln|x| + C$$

Integrate the left dv integral. Solve for v if you can. Replace v with y/x and solve for y if you can.

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3. Linear. $$y' + p(x)y = q(x)$$

Find integrating factor: $\mu = e^{\int p(x)\,dx}$. Multiply both sides by μ.

$$\mu y' + p\mu y = q\mu$$

Apply the Product Rule backwards.

$$(\mu y)' = q\mu$$

Integrate.

$$\mu y = \int q\mu\,dx$$

Solve for y.

$$y(x) = \frac{\int p(x)\mu(x)\,dx + C}{\mu(x)}$$

4. Bernoulli. $$y' + p(x)y = q(x)y^n$$

Use the substitution: $v = y^{1-n}$. Then $y = v^{\frac{1}{1-n}}$. We find y'.

$$y' = \tfrac{1}{1-n} v^{\frac{1}{1-n}-1} \cdot v' = \tfrac{1}{1-n} v^{\frac{n}{1-n}} \cdot v'$$

Substituting these into the original equation gives

$$\frac{1}{1-n} v^{\frac{n}{1-n}}\, v' + p v^{\frac{1}{1-n}} = q v^{\frac{n}{1-n}}$$

Simplify this to get the following.

$$v' + (1-n)p(x)v = (1-n)q(x)$$

This is linear. Solve it, then replace v's with y^{1-n}.

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5. Exact. $$M(x,y) + N(x,y)y' = 0.$$

Test for exactness: does $M_y = N_x$? If yes, proceed. Compute the following integrals.

$$\int M\,dx = \_\_\_\_\_\_ + C_1(y)$$
$$\int N\,dy = \_\_\_\_\_\_ + C_2(x)$$

By inspection find a function in both classes and call it $\psi(x,y)$.

Then the general solution is $\psi(x,y) = C$.

If the original equation was not exact you can try to find an integrating factor or use another method.