Second Order Differential Equations that can be Transformed into First Order Differential Equations

This Lecture covers material developed in the exercises (36-51) on pages 135–136 of the Boyce & DiPrima textbook.

A second order differential equation is one that involves second derivatives. Normally they have two initial conditions.

Example 1. Consider $y'' + \dfrac{y'}{t+1} = 2$ with $y(0) = 1$ and $y'(0) = 2$.

Solution. Notice that there is no y term. If we let $v = y'$ we get

$$v' + \frac{v}{t+1} = 2 \text{ with } v(0) = 2.$$

Now this is a first order equation. It is linear. Let,

$$\mu(t) = e^{\int \frac{1}{t+1}\,dt} = e^{\ln|t+1|+C} = C|t+1|.$$

We will use $\mu = t + 1$. Now we have,

$$\begin{aligned} (t+1)v' + v &= 2(t+1) \\ ((t+1)v)' &= 2t+2 \\ (t+1)v &= t^2+2t+C_1 \\ v &= \frac{t^2+2t+C_1}{t+1} \end{aligned}$$

Since $v(0) = 2$ we get $C_1 = 2$. Now we find $y(t)$.

$$\begin{aligned} y' &= v \\ y &= \int \frac{t^2+2t+2}{t+1}\,dt \\ y &= \int t+1+\frac{1}{t+1}\,dt, \qquad \text{(by long division)} \\ y &= \frac{1}{2}t^2 + t + \ln|t+1| + C_2. \end{aligned}$$

Since $y(0) = 1$ have $1 = 0 + 0 + \ln|1| + C_2$. Hence $C_2 = 1$. Finally,

$$y(t) = \frac{1}{2}t^2 + t + \ln(t+1) + 1, \quad \text{for } t > -1.$$

□

Example 2. Solve $y'y'' = 2$, with $y(0) = 1$ and $y'(0) = 2$.

Solution. Again y does not appear. Let $v = y'$. Then we get $vv' = 2$. This is separable.

$$
\begin{aligned}
\int v\,dv &= \int 2\,dt. \\
\frac{1}{2}v^2 &= 2t + C_1. \\
\frac{1}{2}v^2 &= 2t + 2, \quad \text{since } v(0) = y'(0) = 2. \\
v &= \pm\sqrt{4t+4}. \\
v &= \sqrt{4t+4}, \quad \text{since } v(0) = 2 > 0.
\end{aligned}
$$

Now integrate v to get y.

$$
\begin{aligned}
y &= \int v\,dt \\
&= 2\int \sqrt{t+1}\,dt \\
&= \frac{4}{3}(t+1)^{\frac{3}{2}} + C_2 \\
&= \frac{4}{3}(t+1)^{\frac{3}{2}} - \frac{1}{3}, \quad \text{since } y(0) = 1.
\end{aligned}
$$

Thus,

$$
y(t) = \frac{4(t+1)^{\frac{3}{2}} - 1}{3} \text{ for } t > -1.
$$

□

Example 3. Find the general solution to $yy'' + (y')^2 = 0$, with independent variable t. Notice that t does not appear, except in the differentiation symbols $\frac{d}{dt}$ if we write it out "longhand".

Solution. It turns out the same substitution $v = y'$ will work, but the steps are different. The result at first looks innocent enough:

$$
yv' + v^2 = 0.
$$

It looks separable, but this is not valid. The reason is clearer if we write it as

$$
y\frac{dv}{dt} + v^2 = 0.
$$

There are three variables. The v' did not mean $\frac{dv}{dy}$ so this not an equation form we have studied.

The way around this is to think of v as a function of y. Then

$$
\frac{d\,v(y(t))}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v = v'v,
$$

where v' is understood to mean the derivative with respect to y. Now we have

$$
yvv' + v^2 = 0.
$$

This is now a true first order equation. Dividing by v makes in linear.

$$\begin{aligned} yv' + v &= 0. \\ (yv)' &= 0. \\ yv &= C_1. \\ v &= C_1/y. \end{aligned}$$

Now convert back to y and t.

$$\begin{aligned} \frac{dy}{dt} &= C_1/y. \\ \int y\,dy &= \int C_1\,dt. \\ \frac{1}{2}y^2 &= C_1 t + C_2. \\ y &= \pm\sqrt{C_1 t + C_2}. \end{aligned}$$

This then is the general solution. □

Example 3'. We now consider Example 3 with initial conditions given by $y(t_o) = a$ and $y'(t_o) = b$. If $a \neq 0$, then the sign of a resolves the $\pm$ sign. It is easier to work with

$$y^2 = C_1 t + C_2.$$

Then taking the derivative gives

$$2yy' = C_1.$$

Thus, $C_1 = 2ab$ and $C_2 = a^2 - 2abt_o$.

But, what if $a = 0$? Then we get $C_1 = C_2 = 0$, so the $\pm$ does not matter. It seems $y(t) = 0$. But, this presents a difficultly. What if $y'(t_o) = b \neq 0$? Then there are no solutions. Notice that in Example 3 we divided through by v, which is y'. This step is invalid if y' is ever zero.

Example 4. Find the general solution to $y'' + y(y')^3 = 0$. Let t be the independent variable.

Solution. Let $v = y'$. Then use

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v'v,$$

where the last v' now means the derivative with respect to y. Then the problem becomes,

$$vv' + yv^3 = 0,$$

or

$$v' = -yv^2,$$

which is separable. We have

$$
\begin{aligned}
\int v^{-2}\,dv &= \int -y\,dy \\
-v^{-1} &= -\frac{1}{2}y^2 + C_1 \\
v &= \frac{1}{\frac{1}{2}y^2 - C_1} \\
\frac{dy}{dt} &= \frac{2}{y^2 + C_1}, \quad (\text{new } C_1) \\
\int y^2 + C_1\,dy &= \int 2\,dt \\
\frac{1}{3}y^3 + C_1 y + C_2 &= 2t \\
y^3 + C_1 y + C_2 &= 6t \quad (\text{new } C_1 \& C_2)
\end{aligned}
$$

There is no simple way to solve for y, so we'll leave it in this form.

But, notice that $y = C$ also solves the original differential equation. Can you see why we missed this case? □

Example 4'. Let's look more closely at initial conditions. Suppose $y(t_o) = a$ and $y'(t_o) = b$. This gives the set of two equations in two unknowns.

$$a^3 + C_1 a + C_2 = 6t_o.$$

$$b = \frac{2}{a^2 + C_1/3}.$$

If $b = 0$ this second equation has no solutions. But, in this case the constant function $y(t) = a$ solves the differential equation and satisfies the initial conditions.

Assume $b \neq 0$. Then $C_1 = 3(2/b - a^2)$. From this you can show $C_2 = 6t_o - a^3 - C_1 a$.

Extra Credit. Suppose the initial conditions in Example 4 are $y(0) = a$ and $y(1) = b$. (So, no $y'(0)$.) Show that there are unique values for C_1 and C_2, unless $a = b$. What is the solution when $a = b$?

Example 5. Solve $y'' + (y')^2 - 4y = 2$, with $y(0) = 0$ and $y'(0) = 0$. Let t be the independent variable.

Solution. Let $v = y' = \frac{dy}{dt}$. Then use

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v'v,$$

where the last v' now means the derivative with respect to y. Then the problem becomes,

$$vv' + v^2 - 4y = 2,$$

with y taken as the independent variable. Rewrite as

$$(v^2 - 4y - 2) + vv' = 0.$$

Let $M = v^2 - 4y - 2$ and $N = v$. Then

$$\frac{\partial M}{\partial v} = 2v \qquad \frac{\partial N}{\partial y} = 0.$$

It is not exact, but

$$\frac{M_v - N_y}{N} = 2,$$

which does not depend on v. Thus, we use $\mu = e^{2y}$ as an integrating factor. Now we have

$$e^{2y}(v^2 - 4y - 2) + e^{2y} v v' = 0.$$

You can check that it is exact. Now,

$$\begin{aligned}\psi(y, v) = \int (v^2 - 4y - 2)e^{2y}\, dy &= \frac{1}{2}(v^2 - 2)e^{2y} + (1 - 2y)e^{2y} + C_1(v) \\ &= \frac{1}{2}v^2 e^{2y} - 2ye^{2y} + C_1(v),\end{aligned}$$

and

$$\psi(y, v) = \int v e^{2y}\, dv = \frac{1}{2}v^2 e^{2y} + C_2(y).$$

If we let $C_1(v) = 0$ and $C_2(y) = -2ye^{2y}$ we have our solution:

$$\psi(y, v) = \frac{1}{2}v^2 e^{2y} - 2ye^{2y} = C_3.$$

At $t = 0$ both $y(0)$ and $v(0) = y'(0) = 0$. Thus C_3 must be zero. Now we can simplify and get

$$v^2 = 4y \quad \text{or} \quad (y')^2 = 4y.$$

Thus, $y' = \pm 2\sqrt{y}$, which is separable. Next

$$\int y^{-1/2}\, dy = \pm \int 2\, dt = \pm 2t + C_4.$$

Thus, $2y^{\frac{1}{2}} = \pm 2t + C_4$. Since $y(0) = 0$ we get that $C_4 = 0$. Hence, the solution is

$$y(t) = t^2.$$

□

Summary

In this lecture we have developed two methods for reducing certain second order differential equations to first order differential equations. Both start with a substitution or "change of variables" given by

$$v = \frac{dy}{dt}.$$

Case 1. If the given equation does not contain any "y" terms, replace all occurrences of y' with v and of y'' with v'. In this case the derivatives are all with respect to the original independent variable, t. Once you solve for $v(t)$ integrate it to get $y(t)$. The finial result will have two arbitrary constants. This was the method used in Examples 1 & 2.

Case 2. If the given equation does not contain any "t" terms, the situation is trickier. You replace any occurrences of y' with v, but for y'' you use the following:

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v'v,$$

where v' means the derivative with respect to y. The resulting equation is a first order differential equation in v with independent variable y.

Solve it for $v(y)$. Then you have a problem of the form

$$\frac{dy}{dt} = v(y)$$

which is separable, in fact it is autonomous. Solve it by

$$\int \frac{1}{v(y)}\,dy = \int dt = t + C_2,$$

then solve the result for $y(t)$ if possible.

Finally, check by inspection to see if $y(t) =$ a constant will work. Solutions may not work for all initial conditions.

This was the method used in Examples 3, 4, & 5. (This method could be applied to Example 2 as well, but the first method is usually easier.)

Student Exercises. Solve the following if possible.

(1) $y'' - y' = e^t$, $y(0) = y'(0) = 1$.
(2) $xy'' - 2y' = 4x^3$, $y(0) = 1$, $y'(0) = 3$. Explain what goes wrong.
(3) $y'' + 2yy' = 0$, with t as the independent variable.

Answers.

(1) $y(t) = te^t + 1$.
(2) $y(x) = x^4 + C_1x^3 + C_2$ is the general solution. Why is it not valid for the given initial conditions?
(3) $y = -a\tan(at + b)$ or $a\tanh(at + b)$ or a constant.