## Reduction of Order Method for Finding a Second Solution

This lecture is based on the second half of Section 3.4. It will concern linear second order equations with non-constant coefficients. Suppose we want to find the general solution to

$$y'' + p(t)y' + q(t)y = 0, \qquad (*)$$

and somehow we know $y = f(t)$ is a solution. We need to find another solution that is not a constant multiple of $f(t)$. Here is a trick.

Let $y(t) = v(t)f(t)$. We will plug this into $(*)$ and solve for $v$. We compute the needed derivatives.

$$\begin{aligned} y &= vf \\ y' &= v'f + vf' \\ y'' &= v''f + 2v'f' + vf'' \end{aligned}$$

Plugging into $(*)$ gives

$$v''f + 2v'f' + vf'' + pv'f + pvf' + qvf \;=\; 0.$$

Regrouping the terms gives

$$\begin{aligned} fv'' + (2f' + pf)v' + (f'' + pf' + qf) &= 0, \text{ or} \\ fv'' + (2f' + pf)v' &= 0 \text{ (since } f \text{ is a solution).} \end{aligned}$$

Next let $w = v'$ and write

$$fw' + (2f' + pf)w = 0. \qquad (@)$$

Thus, we have reduced the order from two to one. Notice that the resulting equation is both linear and separable. Solve it for $w$, then integrate $w$ to get $v$ and hope it is not a constant.

**Example 1.** Consider

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0. \qquad (\#)$$

assume $x > 1$. After staring at it for a while we notice $y_1(x) = x$ is a solution. Check this. Let $y = y_2(x) = v(x)x$. Then $y = vx$, $y' = v'x + v$, and $y'' = v''x + 2v'$. Thus (#) becomes

$$v''x + 2v' - \frac{x}{x-1}(v'x + v) + \frac{1}{x-1}vx = 0,$$

or

$$xv'' + \left(2 - \frac{x^2}{x-1}\right)v' + \left(\frac{-x}{x-1} + \frac{x}{x-1}\right)v = 0.$$

Thus we have

$$v'' + \left(\frac{2}{x} - \frac{x}{x-1}\right)v' = 0.$$

Let $w = v'$. Then we have

$$\frac{dw}{dx} + \left(\frac{2}{x} - \frac{x}{x-1}\right)w = 0.$$

We will use that it is separable to get

$$\int \frac{1}{w}\,dw = \int \left(\frac{x}{x-1} - \frac{2}{x}\right)\,dx.$$

Thus, since $x/(x-1) = 1 + 1/(x-1)$, we have

$$\ln|w| = x + \ln(x-1) - 2\ln x + C.$$

Therefore,

$$w = Ce^x\left(\frac{x-1}{x^2}\right).$$

Since we do not need the general solution for $v$ we let $C = 1$.

We integrate $w$ to get $v$,

$$\begin{aligned} v &= \int e^x\left(\frac{x-1}{x^2}\right)\,dx \\ &= \int \frac{e^x}{x}\,dx - \int \frac{e^x}{x^2}\,dx \end{aligned}$$

We use integration by parts on the second integral. Let $u = e^x$ and $dz = x^{-2}dx$. Then $du = e^x dx$ and $z = -x^{-1}$. Thus,

$$\int \frac{e^x}{x^2} = -\frac{e^x}{x} + \int \frac{e^x}{x}\, dx$$

Thus,

$$v = \int \frac{e^x}{x}\, dx + \frac{e^x}{x} - \int \frac{e^x}{x}\, dx = \frac{e^x}{x}.$$

Finally,

$$y_2(x) = \frac{e^x}{x} \cdot x = e^x,$$

is our second solution. Plug it into (#) and check this! The general solution is

$$y(x) = C_1 x + C_2 e^x.$$

So there!

One last point. Could it happen that we go through all this work and turns out that $v$ is a constant, so that we do not get a new solution that is not a multiple of the first? No. Because if $v$ is a constant then $w = 0$. But we know that (@) has nontrivial solutions since otherwise solutions for many valid initial value problems would not exist.

**Exercise.** Suppose in Example 1 we were given that $y_1 = e^x$ was a solution. Use this method to show that $y_2 = x$ is another solution that is not a constant multiple of $y_1$.