Homogeneous Second Order Differential Equations with Constant Coefficients: Continued

Complex roots.

The final case is what to do when the roots of the characteristic polynomial are complex. Recall that this means they will be of the form $\alpha \pm \beta i$ for real numbers α and β where $i^2 = -1$. (This assumes that the coefficients a, b and c are real.) We will need to "review" some facts about complex functions that where censored from your calculus textbook.

But first, let's look at a simple example, $y'' + y = 0$. We can rewrite this as $y'' = -y$. So, we are seeking functions whose second derivatives are their own negatives. Two might come to mind, $\sin x$ and $\cos x$. In fact $y = C_1 \sin x + C_2 \cos x$ gives all possible solutions, as we will show later.

The roots of the characteristic polynomial, $r^2 + 1 = 0$, are $\pm i$. Notice that

$$(e^{ix})'' = (ie^{ix})' = i^2 e^{ix} = -e^{ix}.$$

But, what does it mean to raise e to a complex power? And, what does it mean to take a derivative of such a function? And, how are these functions connected to $\sin x$ and $\cos x$?

Way back in Calculus II you studied Taylor series and you learned that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

Suppose $z = a + ib$ is a complex number. Then we define

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots .$$

In courses on Complex Analysis (MATH 455 here) it is shown that this sequence converges for all complex numbers z. The derivative can be defined via term-by-term differentiation. The following facts can also be proven:

$$e^{a+ib} = e^a e^{ib}$$

$$\frac{de^{cx}}{dx} = ce^{cx},$$

for any complex number c and x is a real variable.

Now watch.

$$e^{ix} = 1+ix+\frac{(ix)^2}{2}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\frac{(ix)^6}{6!}+\frac{(ix)^7}{7!}+\frac{(ix)^8}{8!}+\cdots$$
$$= 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$$
$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$
$$= \cos x + i \sin x.$$

Now let's get back to differential equations. Suppose we have $ay'' + by' + cy = 0$ and the roots of $ar^2 + br + c$ are $r = \alpha \pm i\beta$. Then the general solution is

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = (C_1 e^{i\beta x} + C_2 e^{-i\beta x})e^{\alpha x}$$
$$= (C_1(\cos(\beta x) + i\sin(\beta x)) + C_2(\cos(\beta x) - i\sin(\beta x)))e^{\alpha x}$$
$$= ((C_1 + C_2)\cos(\beta x) + i(C_1 - C_2)\sin(\beta x))\, e^{\alpha x}.$$

We can rewrite this as

$$Ae^{\alpha x}\cos\beta x + Be^{\alpha x}\sin\beta x.$$

Theorem 3. The general solution to $ay'' + by' + cy = 0$ when the roots of the characteristic polynomial are $\alpha \pm i\beta$ is

$$y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x.$$

If $y(x_0) = p$ and $y'(x_0) = q$ then there is a unique solution for C_1 and C_2; they will be real as long as a, b, c, p and q are real.

Proof. We have already derived the solution, but you can check it by directly substituting it in to the differential equation. Next, $y(x_0) = p$ implies

$$C_1 \cos\beta x_0 + C_2 \sin\beta x_0 = pe^{-\alpha x_0}.$$

And $y'(x_0) = q$ implies

$$C_1 \alpha e^{\alpha x_0} \cos \beta x_0 - C_1 \beta e^{\alpha x_0} \sin \beta x_0 + C_2 \alpha e^{\alpha x_0} \sin \beta x_0 + C_2 \beta e^{\alpha x_0} \cos \beta x_0 = q,$$

or

$$C_1(\alpha \cos \beta x_0 - \beta \sin \beta x_0) + C_2(\beta \cos \beta x_0 + \alpha \sin \beta x_0) = q e^{-\alpha x_0}.$$

So, again we have two equations and two unknowns and these can readily by solved for C_1 and C_2. □

Example. Find the general solution to $y'' - y' + 2y = 0$. Then find the solution for the initial values $y(0) = p$, $y'(0) = q$.

Solution. The characteristic polynomial $r^2 - r + 2$ has complex roots $r = \frac{1}{2} \pm i\frac{\sqrt{7}}{2}$. Thus, the general solution is

$$y(x) = Ae^{\frac{1}{2}x} \cos\left(\frac{\sqrt{7}}{2}x\right) + Be^{\frac{1}{2}x} \sin\left(\frac{\sqrt{7}}{2}x\right).$$

Now, $y(0) = p$ implies $A = p$ and $y'(0) = q$ gives $p/2 + B\sqrt{7}/2 = q$. Thus, $B = \frac{2q-p}{\sqrt{7}}$ and we have

$$y(x) = pe^{\frac{1}{2}x} \cos\left(\frac{\sqrt{7}}{2}x\right) + \frac{2q-p}{\sqrt{7}} e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{7}}{2}x\right).$$

□

Summary

Given $ay''(x) + by'(x) + cy(x) = 0$ we have three cases. These depend on the roots of the characteristic polynomial $ar^2 + br + c = 0$.

Case 1. The roots of the characteristic polynomial, r_1 and r_2, are real and distinct, that is $r_1 \neq r_2$. Then the general solution is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$

Case 2. The characteristic polynomial has a single real root, r. Then the general solution is

$$y(x) = C_1 e^{rx} + C_2 x e^{rx}.$$

Case 3. The roots of the characteristic polynomial are complex conjugates, $\alpha \pm \beta i$. Then the general solution is

$$y(x) = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x.$$

In each case we can find unique values of C_1 and C_2 for any given pair of initial conditions of the form

$$y(x_0) = p \quad \& \quad y'(x_0) = q.$$