Homogeneous Second Order Differential Equations with Constant Coefficients

This Lecture covers material in Sections 3.1, 3.4 & 3.3, in that order. First a quick outline. A **second order linear differential equation with constant coefficients** is a equation of the form

$$ay'' + by' + cy = g(x).$$

It is second order because it involves the second derivative and no higher derivatives. It is linear because y'', y' and y are linear. The a, b and c are constants, rather than functions of x. If $g(x) = 0$ (for all x) then the equation is said to be **homogeneous**. For now we restrict studies to this case. Associated to each second order linear differential equation with constant coefficients is a **characteristic polynomial**

$$ar^2 + br + c.$$

If the roots of the characteristic polynomial are real and distinct, the method in 3.1 works to find all solutions to $ay'' + by' + cy = 0$. If the characteristic polynomial has a repeated real root, the method in 3.4 works. If it has a pair of complex roots the method in 3.3 works.

Note: The second substitution method discussed in the previous lecture can be used for these homogeneous equations, but it is much harder and should only be used when easier methods fail.

Section 3.2 deals with some theoretical matters and we shall return it after we have done 3.1, 3.4 and 3.3. Sections 3.5 & 3.6 deal with nonhomogeneous systems, that is when $g(x)$ is not the zero function. Sections 3.7 & 3.8 deal with applications of second order systems.

Distinct Real Roots.

We are basically going to take a guess and check approach. We know equations of the form $ay'+by=0$ have general solution $y(x) = Ce^{-\frac{b}{a}x}$. Notice that $-b/a$ is the root of $ar+b=0$. Consider

$$ay'' + by' + cy = 0. \qquad (*)$$

Let's just suppose $y(x) = e^{rx}$ is a solution. If we plug it in to $(*)$ we get

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Divide through by e^{rx}, which can never be zero, to get

$$ar^2 + br + c = 0.$$

But this is just the characteristic polynomial set equal to zero. This is also called the **characteristic equation** of $(*)$. If r is a root of the characteristic polynomial then $y(x) = e^{rx}$ is a solution.

Example 1. Find two solutions to $y'' - 4y' + 3y = 0$.

Solution. The characteristic equation is $r^2-4r+3 = (r-3)(r-1) = 0$. The roots are 1 and 3. Therefore, $y_1 = e^x$ and $y_2 = e^{3x}$ should be solutions. Let's check.

$$(e^x)'' - 4(e^x)' + 3e^x = e^x - 4e^x + 3e^x = (1-4+3)e^x = 0.\checkmark$$

$$(e^{3x})'' - 4(e^{3x})' + 3e^{3x} = 9e^{3x} - 12e^{3x} + 3e^{3x} = (9-12+3)e^{3x} = 0.\checkmark$$

It should be clear that C_1e^x and C_2e^{3x} are also solutions. Just plug them in. Less obvious is that any function of the form $y = C_1e^x + C_2e^{3x}$ is a solution. Let's check.

$$
\begin{aligned}
y'' - 4y' + 3y &= (C_1e^x + C_2e^{3x})'' - 4(C_1e^x + C_2e^{3x})' + 3(C_1e^x + C_2e^{3x}) \\
&= C_1e^x + 9C_2e^{3x} - 4C_1e^x - 12C_2e^{3x} + 3C_1e^x + 3C_2e^{3x} \\
&= C_1(1 - 4 + 3)e^x + C_2(9 - 12 + 3)e^{3x} \\
&= 0 + 0 = 0. \checkmark\checkmark
\end{aligned}
$$

So, we have done even better than what we first asked for; we have a two parameter infinite family of solutions. We will see in Section 3.2 that this is the complete solution set, or in other words, we have found the *general solution.* $\square$

Definition. For a homogeneous second order linear differential equation, a pair of solutions in which neither is a multiple of the other other is called a **fundamental solution set**. Thus, in Example 1, $\{e^x, e^{3x}\}$ is a fundamental solution set. Its members generate all other solutions as we shall show later.

We know that when first order differential equations are used in applied problems there was usually an initial condition. With second order differential equations the general solution will have two arbitrary constants and two initial conditions are required to resolve them. They are usually of the form $y(x_0) = y_0$ and $y'(x_0) = y'_0$.

Example 2. For the equation in Example 1, suppose we were given $y(0) = 1$ and $y'(0) = 2$. Find values of C_1 and C_2 that give the particular solution in this case.

Solution. We had $y(x) = C_1e^x + C_2e^{3x}$. Since $y(0) = 1$, we have $C_1 + C_2 = 1$. Now

$$y'(x) = C_1e^x + 3C_2e^{3x}.$$

Thus $y'(0) = 2$ implies $C_1+3C_2 = 2$. So, we have two equations and two unknowns. You can solve these to get $C_1 = C_2 = 1/2$. Thus, the solution is

$$y(x) = \frac{e^x + e^{3x}}{2}.$$

□

Theorem 1. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has two real distinct roots, r_1 and r_2. Then $y = C_1e^{r_1x} + C_2e^{r_2x}$ is a solution and for any initial conditions of the form $y(x_0) = p$ and $y'(x_0) = q$ there is a unique choice of C_1 and C_2 that solve it. In other words $\{e^{r_1x}, e^{r_2x}\}$ is a fundamental solution set.

Proof. For the first part we just plug in.

$$\begin{aligned} ay'' + by' + cy &= a(C_1e^{r_1x} + C_2e^{r_2x})'' + b(C_1e^{r_1x} + C_2e^{r_2x})' \\ &\qquad + c(C_1e^{r_1x} + C_2e^{r_2x}) \\ &= aC_1r_1^2e^{r_1x} + aC_2r_2^2e^{r_2x} + bC_1r_1e^{r_1x} + bC_2r_2e^{r_2x} \\ &\qquad + cC_1e^{r_1x} + cC_2e^{r_2x} \\ &= C_1(ar_1^2 + br_1 + c)e^{r_1x} + C_2(ar_2^2 + br_2 + c)e^{r_2x} \\ &= C_1 \cdot 0 \cdot e^{r_1x} + C_2 \cdot 0 \cdot e^{r_2} = 0. \end{aligned}$$

For the initial conditions we get two equations.

$$y(x_0) = p \implies C_1 e^{r_1 x_0} + C_2 e^{r_2 x_0} = p$$

and

$$y'(x_0) = q \implies C_1 r_1 e^{r_1 x_0} + C_2 r_2 e^{r_2 x_0} = q.$$

Therefore,

$$C_1 = \frac{r_2 p - q}{r_2 - r_1} e^{-r_1 x_0} \;\; \& \;\; C_2 = \frac{r_1 p - q}{r_1 - r_2} e^{-r_2 x_0}.$$

Note that this would fail if $r_1 = r_2$. □

Repeated Root.

Next we look at an example with a repeated root.

Example 3. Find the general solution to $y'' + 4y' + 4y = 0$. Find particular solutions for (a) $y(0) = 3$, $y'(0) = -6$, and for (b) $y(0) = 3$, $y'(0) = 1$.

Solution. The characteristic equation is $r^2 + 4r + 4 = 0$. The only root is $r = -2$. You can check that $y = Ce^{-2x}$ is a solution for any value of C.

(a) $y(0) = 3$ implies $C = 3$. This value for C also works for $y'(0) = -6$, since $y'(x) = -6e^{-2x}$.

(b) $y(0) = 3$ implies $C = 3$, but $y'(0) = 1$ implies $C = -1/2$. Both cannot be true, so we cannot find a solution for these initial conditions.

What to do? We need a second solution that is not a multiple of e^{-2x}, but whose derivatives behave similarly to those of e^{-2x}. Perhaps after many false starts, someone guessed $y = xe^{-2x}$. Let's try it. We compute $y' = e^{-2x} - 2xe^{-2x}$ and $y'' = -2e^{-2x} - 2e^{-2x} +$

$4xe^{-2x} = -4x^{-2x} + 4xe^{-2x}$. Plugging into the original differential equation gives

$$y'' + 4y' + 4y = (-4x^{-2x} + 4xe^{-2x}) + 4(e^{-2x} - 2xe^{-2x}) + 4(xe^{-2x})$$

$$= (-4+4)e^{-2x} + (4-8+4)xe^{-2x}$$

$$= 0 + 0 = 0.\checkmark$$

Now, let $y = C_1e^{-2x} + C_2xe^{-2x}$. We leave it to you to plug this into $y'' + 4y' + 4y$ and show it is zero for all choices of C_1 and C_2. We will use it to solve the initial value problem (b). Will still have $y(0) = 3$ implies $C_1 = 3$. Next

$$y'(x) = (3e^{-2x} + C_2xe^{-2x})' = -6e^{-2x} + C_2e^{-2x} - 2C_2xe^{-2x}.$$

Thus, $y'(0) = -6 + C_2 = 1$ implies $C_2 = 7$. Thus, our solution is

$$y(x) = 3e^{-2x} + 7xe^{-2x}.$$

□

But, maybe that was just dumb luck? No.

Theorem 2. Suppose the characteristic polynomial of $ay''+by'+cy = 0$ has a single real root, r. Then $y = C_1e^{rx}+C_2xe^{rx}$ is a solution and for any initial conditions of the form $y(x_o) = p$ and $y'(x_0) = q$ there is a unique choice of C_1 and C_2 that solves it. In other words $\{e^{rx}, xe^{rx}\}$ is fundamental solution set.

Solution. Again, we just plug in. But, remember by the quadratic formula we have that $r = -b/2a$.

$$\begin{aligned} ay'' + by' + cy &= a(C_1e^{rx} + C_2xe^{rx})'' + b(C_1e^{rx} + C_2xe^{rx})' \\ &\qquad + c(C_1e^{rx} + C_2xe^{rx}) \\ &= aC_1r^2e^{rx} + 2aC_2re^{rx} + aC_2r^2xe^{rx} + bC_1re^{rx} + bC_2e^{rx} \\ &\qquad + bC_2rxe^{rx} + cC_1e^{rx} + cC_2xe^{rx} \\ &= C_1(ar^2 + br + c)e^{rx} + C_2(2ar + b)e^{rx} \\ &\qquad + C_2(ar^2 + br + c)xe^{rx} \\ &= 0 + 0 + 0 = 0. \end{aligned}$$

Now for the initial conditions.

$$y(x_0) = p \implies C_1e^{rx_0} + C_2x_0e^{rx_0} = p$$

and

$$y'(x_0) = q \implies C_1re^{rx_0} + C_2e^{rx_0} + C_2rx_0e^{rx} = q.$$

It is still two equations and two unknowns. Multiply the first by r and subtract the second equation to isolate C_2.

$$C_2 = \frac{rp - q}{rx_0 - 1 - rx_0}e^{-rx_0} = (q - rp)e^{-rx_0}.$$

Then solving for C_1 gives

$$C_1 = pe^{-rx_0} - C_2x_0 = ((1 + rx_0)p - x_0q)e^{-rx_0}.$$

□

Additional Examples.

Before going on to the case with complex roots, we do a few examples and analyze the results. (I used t for the independent variable instead of x.)

Example 4. Find the general solution to $y'' - 2y' = 0$.

Solution. The characteristic equation is $r^2 - 2r = r(r-2) = 0$. Thus, the fundamental solution set is $\{e^{0t}, e^{2t}\} = \{1, e^{2t}\}$. Hence the general solution is

$$y = C_1 + C_2 e^{2t}.$$

$\square$

Example 5. Find the solution to $y'' = 0$, with $y(0) = 2$ and $y'(0) = 5$.

Solution. You know from Calculus I that the general solution is $y = C_1 t + C_2$. Let's see what we get using the methods developed here. The characteristic equation is $r^2 = 0$. The fundamental solution set is $\{e^{0t}, te^{0t}\} = \{1, t\}$. Therefore, the general solution is

$$y = C_1 t + C_2.$$

You can check that the particular solution is $y = 5t + 2$. $\square$

Example 6. Find the solution to $y'' + 5y' + 6y = 0$, with $y(0) = 0$ and $y'(0) = 1$. Then find the maximum of $y(t)$ for $t \geq 0$.

Solution. The characteristic equation is $r^2 + 5r + 6 = (r+2)(r+3) = 0$. Thus the general solution is

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t}.$$

The initial conditions give

$$C_1 + C_2 = 0 \qquad \& \qquad -2C_1 - 3C_2 = 1.$$

Solving these gives $C_1 = 1$ and $C_2 = -1$. Thus,

$$y(t) = e^{-2t} - e^{-3t}.$$

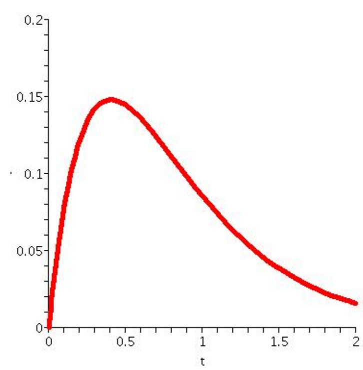
To find the maximum we compute $y'(t)$ and set it equal to zero and solve for t.

$$\begin{aligned} y'(t) &= -2e^{-2t} + 3e^{-3t} = 0 \\ 0 &= -2e^{t} + 3 \quad \text{(multiply by } e^{3t}) \\ e^{t} &= \frac{3}{2} \\ t &= \ln\frac{3}{2} \approx 0.405465108 \end{aligned}$$

Thus the maximum is

$$y(\ln\frac{3}{2}) = \frac{4}{9} - \frac{8}{27} = \frac{4}{27} = 0.148148148...$$

See the graph below. □



Example 7. An important consideration in many problems is whether or not the solution is bounded. Consider $y''+(1+\sigma)y'+\sigma y = 0$. For which values of the parameter σ are we assured that the solution is bounded for positive values of t?

Solution. The characteristic equation is

$$r^2 + (1+\sigma)r + \sigma = (r+1)(r+\sigma) = 0.$$

If $\sigma \neq 1$ the general solution is

$$y(t) = C_1 e^{-t} + C_2 e^{\sigma t}.$$

It is clear that if $\sigma \leq 0$ the solution is bounded. If $\sigma > 0$ the solution would only be bounded if the initial conditions gave $C_2 = 0$.

If $\sigma = 1$ then the general solution is

$$y(t) = C_1 e^{-t} + C_2 t e^{-t}$$

This is bounded for $t > 0$. (Use L'Hospital's Rule to show that the limit as $t \to \infty$ is always zero.) □

Example 8. In Chapter 4 we will study higher order differential equations. For now we wet your appetite with some examples. Find the general solution to each of the following.

(a) $2y''' + y'' - 5y' - 6y = 0$.
(b) $y''' - 3y' - 2 = 0$.
(c) $y''' - 3y'' + 3y - 1 = 0$.

Solutions. (a) The characteristic polynomial is

$$2r^3 + r^2 - 5r - 6 = (r+1)(r-2)(2r+3)$$

Its roots are $r = -3/2$, -1 and 2. They are real and distinct. The general solution is

$$y(t) = C_1 e^{-3t/2} + C_2 e^{-t} + C_3 e^{2t}.$$

(b) The characteristic polynomial is

$$r^3 - 3r - 2 = (r+1)^2(r-2).$$

Its roots are $r = -1$ and 2. They are real, but -1 is repeated. The general solution is

$$y(t) = C_1 e^{-t} + C_2 t e^{-t} + C_3 e^{2t}.$$

(c) The characteristic polynomial is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3.$$

Its only root is $r = 1$, with multiplicity three. The general solution is

$$y(t) = C_1 e^t + C_2 t e^t + C_3 t^2 e^t.$$

You can plug these in and check. Can you see the pattern?

$\square$