Nonhomogeneous Second Order Equations with Constant Coefficients

We will be studying equations of the form

$$ay''(x) + by'(x) + cy(x) = g(x),$$

where $g(x)$ is continuous on some open interval of interest and a, b and c are real constants, $a \neq 0$. We may refer to $g(x)$ as a forcing term for reasons that will become clear when we study applications to mass-spring systems.

- In Section 3.5 we develop the method of **Undetermined Coefficients.** It is relatively easy, but does not always apply.
- In Section 3.6 we develop the method of **Variation of Parameters.** It is harder, but always works.

First we make some general remarks about solutions to nonhomogeneous second order equations with constant coefficients. For the homogeneous equations of this type we had the useful fact that linear combinations of solutions were solutions. That is, if $y_1(x)$ and $y_2(x)$ both solved

$$ay'' + by' + cy = 0,$$

then $y = C_1y_1 + C_2y_2$ was also a solution. This does not hold for nonhomogeneous equations. For example, $f(x) = (x - 1)/4$ is a solution to

$$y'' + 4y' + 4y = x, \qquad (*)$$

as you can check. But, $4f(x)$ is not a solution. We know that e^{-2x} and xe^{-2x} are solutions to the homogeneous problem

$$y'' + 4y' + 4y = 0.$$

Now consider $y(x) = C_1e^{-2x} + C_2xe^{-2x} + (x-1)/4$. If you plug this into $(*)$ you will see that it is a solution. In fact, it is the general solution.

Theorem 1. *Any solution to*

$$ay'' + by' + cy = g(x)$$

can be written in the form

$$C_1 y_1 + C_2 y_2 + y_p$$

where $\{y_1, y_2\}$ *is a fundamental solution set for the corresponding homogeneous equation*

$$ay'' + by' + cy = 0$$

and y_p *is any particular solution of the original equation.*

Proof. It is easy to see why such functions are solutions. Let $y = C_1 y_1 + C_2 y_2 + y_p$ and plug it into the original nonhomogeneous equation.

$$ay'' + by' + cy =$$

$$a(C_1 y_1 + C_2 y_2 + y_p)'' + b(C_1 y_1 + C_2 y_2 + y_p)' + c(C_1 y_1 + C_2 y_2 + y_p) =$$

$$a(C_1 y_1 + C_2 y_2)'' + a y_p'' + b(C_1 y_1 + C_2 y_2) + b y_p' + c(C_1 y_1 + C_2 y_2) + c y_p =$$

$$\big[a(C_1 y_1 + C_2 y_2)'' + b(C_1 y_1 + C_2 y_2) + c(C_1 y_1 + C_2 y_2)\big] + \big[a y_p'' + b y_p' + c y_p\big]$$

$$= 0 + g(x) = g(x).$$

Now, suppose $f(x)$ is any other solution of the original equation. Consider $h(x) = f(x) - y_p(x)$. Then

$$ah'' + bh' + ch = [af'' + bf' + cf] - [y_p'' + by_p' + cy_p] = g - g = 0.$$

Therefore, the difference between any two solutions of the nonhomogeneous equation is a solution of the corresponding homogeneous equation. But we know any solution of the homogeneous equation is of the form $C_1y_1 + C_2y_2$. Thus,

$$h(x) = C_1y_1 + C_2y_2,$$

for some C_1 and C_2. Therefore, we can write

$$f(x) = C_1y_1(x) + C_2y_2(x) + y_p(x).$$

$\square$

Corollary. Suppose $y = y_f$ is a solution to $ay'' + by' + cy = f$ and $y = y_g$ is a solution to $ay'' + by' + cy = g$. Then $y = y_f + y_g$ is a solution to $ay'' + by' + cy = f(x) + g(x)$.

The proof is easy and left to you.

1. The Method of Undermined Coefficients

We start with some simple examples with first degree equations.

Example 1. Find the general solution to $y' = xe^x$.

Solution. Of course, this is the same as finding the indefinite integral $\int xe^x\,dx$. But, let's suppose you are too lazy to look up integration by parts. You remember however that this integral of the form $(Ax + B)e^x$ plus an arbitrary constant. If so then

$$\big((Ax + B)e^x\big)' = xe^x.$$

Hence,

$$Ae^x + (Ax + B)e^x = Axe^x + (A + B)e^x = xe^x.$$

Since e^x and xe^x are linearly independent we know that $A = 1$ and $B = -1$ is the only solution. Hence,

$$y = (x - 1)e^x + C.$$

Example 2. Find the general solution to $y' = x^3e^x$.

Solution. You could do integration by parts three times, or you might make an educated guess that

$$\int x^3e^x\,dx = (Ax^3 + Bx^2 + Cx + D)e^x + \text{arbitrary constant}.$$

Then

$$(3Ax^2 + 2Bx + C)e^x + (Ax^3 + Bx^2 + Cx + D)e^x = x^3e^x.$$

Regrouping terms gives

$$Ax^3e^x + (3A + B)x^2e^x + (2B + C)xe^x + (C + D)e^x = x^3e^x.$$

Since $\{x^3e^x, x^2e^x, xe^x, e^x\}$ is linearly independent (see chapter 4 for how to check this claim), we know that

$$A = 1 \quad B = -1/3 \quad C = 2/3\ \&\ D = -2/3.$$

Therefore,

$$y = \int x^3 e^x \, dx = \left(x^3 - \frac{1}{3}x^2 + \frac{2}{3}xe^x - \frac{2}{3} \right) e^x + \text{arbitrary constant.}$$

□

Student Exercise 1. Find the general solution to

$$y' = 2x^2 e^x + 3xe^x + e^x.$$

Check your answer by computing its derivative.

1.1. **Exponential Forcing Terms.**

Now we start looking at second degree equations.

Example 3. Find the general solution to $y'' + y = e^x$.

Solution. We know that the solution to the corresponding homogeneous equation is

$$y_h = C_1 \cos x + C_2 \sin x.$$

Motivated by the first two examples we guess that something of the form

$$y_p = Ae^x$$

might be a particular solution to the original problem. Let's check.

$$y_p'' + y_p = 2Ae^x.$$

We need for this to equal e^x. Therefore, $A = 1/2$. We conclude that the general solution is

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{2}e^x.$$

□

We might conjecture that given $ay'' + by' + cy = de^{\alpha x}$ the general solution will be $y = C_1 y_1 + C_2 y_2 + Ae^{\alpha x}$, where $\{y_1, y_2\}$ is a fundamental solution set for the corresponding homogeneous problem, C_1 and C_2 are arbitrary constants, and A is to be determined. This almost works. It fails if $e^{\alpha x}$ happens to be a solution to the homogeneous problem as in the next example.

Example 4. Find the general solution to $y'' - y' - 2y = 3e^{2x}$.

Wrong Solution. We know that the solution to the corresponding homogeneous equation is

$$y_h = C_1 e^{2x} + C_2 e^{-x}.$$

Let $y_p = Ae^{2x}$. We plug in a solve for A. Thus, we need for

$$4Ae^{2x} - 2Ae^{2x} - 2Ae^{2x} = 3e^{2x}.$$

But then we get $0A = 3$. Yikes! But, of course this had to happen since e^{2x} is a solution to the homogeneous problems. What to do? □

Right Solution. We need something whose derivatives behave kind of like e^{2x} but which is linearly independent from e^{2x}. When we encountered something like this before, in the repeated roots case, we ended up using xe^{2x}. Let's try $y_p = Axe^{2x}$ and see what happens. Now, $y_p' = Ae^{2x} + 2Axe^{2x}$ and $y_p'' = 4Ae^{2x} + 4Axe^{2x}$. Thus, we need for

$$4Ae^{2x} + 4Axe^{2x} - Ae^{2x} - 2Axe^{2x} - 2Axe^{2x} = 3e^{2x}.$$

Simplifying gives

$$8Ae^{2x} = 3e^{2x}.$$

Hence $A = 3/8$ and we conclude that the general solution is

$$y = C_1 e^{2x} + C_2 e^{-x} + \tfrac{3}{8} x e^{2x}.$$

□

Example 5. Find the general solution to $y'' - 4y' + 4y = e^{2x}$.

Solution. We know that the solution to the corresponding homogeneous equation is

$$y_h = C_1 x^{2x} + C_2 x e^{2x}.$$

Now, both e^{2x} and xe^{2x} are solutions to the homogeneous problem and so y_p cannot be multiples of these. Instead we try $y_p = Ax^2e^{2x}$. It cannot hurt to try it. Now $y_p' = 2Ax^2e^{2x} + 2Axe^{2x}$ and $y_p'' = 4Ax^2e^{2x} + 8Axe^{2x} + 2Ae^{2x}$. Thus, we need to solve

$$4Ax^2e^{2x} + 8Axe^{2x} + 2Ae^{2x} - 8Ax^2e^{2x} - 8Axe^{2x} + 4Ax^2e^{2x} = e^{2x}.$$

Simplifying gives

$$2A = 1.$$

Thus, the general solution is

$$y = C_1x^{2x} + C_2xe^{2x} + \tfrac{1}{2}x^2e^{2x}.$$

□

It could be that I just picked examples where these tricks worked. But no, there is a general theorem that backs this up.

Theorem 2. *Suppose, $ay'' + by' + cy = de^{\alpha x}$. Then we have three cases.*

(1) *If $e^{\alpha x}$ is not a solution of the corresponding homogeneous problem, then the general solution is*

$$y = C_1y_1 + C_2y_2 + Ae^{2x},$$

where $\{y_1, y_2\}$ is a fundamental solution set of the homogeneous problem and A is determined by substitution.

(2) *If $e^{\alpha x}$ is a solution of the corresponding homogeneous problem, and $xe^{\alpha x}$ is not, then the general solution is*

$$y = C_1y_1 + C_2y_2 + Axe^{2x},$$

where $\{y_1, y_2\}$ is a fundamental solution set of the homogeneous problem and A is determined by substitution.

(3) *If* $e^{\alpha x}$ *and* $xe^{\alpha x}$ *are solutions of the corresponding homogeneous problem, then the general solution is*

$$y = C_1 y_1 + C_2 y_2 + Ax^2 e^{2x},$$

where $\{y_1, y_2\}$ *is a fundamental solution set of the homogeneous problem and* A *is determined by substitution.*

Proof. (1) Let $y_p = Ae^x$. Then $y_p' = \alpha A e^{\alpha} x$ and $y_p'' = \alpha^2 A e^{\alpha x}$. We plug into the given differential equation to get

$$(a\alpha^2 + b\alpha + c)Ae^{\alpha x} = de^{\alpha x}.$$

Since in case (1) α is not a root of the characteristic polynomial, we know the expression in parenthesis above in not zero. Hence

$$A = \frac{d}{a\alpha^2 + b\alpha + c}$$

makes y_p a solution.

(2) Let $y_p = Axe^{\alpha x}$. Then $y_p' = Ae^{\alpha x} + \alpha A x e^{\alpha x}$ and $y_p'' = 2\alpha A e^{\alpha x} + \alpha^2 A x e^{\alpha x}$. We plug into the given differential equation to get

$$(a2\alpha + a\alpha^2 x + b + b\alpha x + cx)Ae^{\alpha x} =$$

$$\left((a2\alpha + b) + \overbrace{a\alpha^2 + b\alpha + c}^{0}\, x\right) Ae^{\alpha x} = (a2\alpha + b)Ae^{\alpha x}.$$

Now we need to solve

$$(2a\alpha + b)Ae^{\alpha x} = de^{\alpha x}.$$

But since α is NOT a double root of the characteristic polynomial we know that $\alpha \neq -b/2a$ and hence $2a\alpha + b \neq 0$. Thus,

$$A = \frac{d}{2a\alpha + b}.$$

(3) Let $y_p = Ax^2 e^{\alpha x}$. The reader can and will check that substitution into the original differential equation gives

$$\left(2a + 2\overbrace{(2a\alpha + b)}^{0}x + \overbrace{(a\alpha^2 + b\alpha + c)}^{0}x^2 \right) Ae^{\alpha x} = de^{\alpha x}.$$

Thus, $A = d/2a$. □

Student Exercises 2. Find the general solution to each of the following.

a. $y'' + y' - 6y = 3e^{2x} - e^x$.
b. $y'' + 4y' + 4y = 2e^{3x} + e^{-2x}$.
c. $y'' - y' = e^x + 2$.

1.2. **Polynomial Forcing Terms.** Next we look at another family of examples. These are of the form $ay'' + by' + cy = P(x)$, where $P(x)$ is a polynomial.

Example 6. Find the general solution to $y'' + y = 2x^2 + x$.

Solution. We know that the solution to the corresponding homogeneous equation is

$$y_h = C_1 \cos x + C_2 \sin x.$$

Now, what to try for y_p? We might guess that since, in a sense, "two integrations" are involved, we should try $y_p = Ax^4 + Bx^3 + Cx^2 + Dx + E$. That means we have to find five coefficients. But, we realize that terms involving x^4 and x^3 cannot play any role, because there will not be any such terms in y_p'' to cancel them out. So, we try

$$y_p = Ax^2 + Bx + C.$$

Then, we get

$$y_p'' + y_p = 2A + Ax^2 + Bx + C = 2x^2 + x.$$

Thus, $A = 2$, $B = 1$ and $C = -2A = -4$. Therefore, the general solution is

$$y = C_1 \cos x + C_2 \sin x + 2x^2 + x - 4.$$

□

Theorem 3. *Suppose $ay'' + by' + cy = P(x)$, where $P(x)$ is a polynomial of degree n and $c \neq 0$. Let*

$$y_p = A_n x^n + \cdots A_1 x + A_0.$$

Then we can find values for the A_i's that give a solution.

Proof. Let $P(x) = P_n x^n + P_{n-1}x^{n-1} + \cdots + P_1 x + P_0$. Then substituting y_p into the given differential equation gives

$$a\big(n(n-1)A_n x^{n-2} + \cdots + 6A_3 x + 2A_2\big) + b\big(nA_n x^{n-1} + \cdots + 2A_2 x + A_1\big)$$

$$+c(A_n x^n + \cdots + A_1 x + A_0) = P(x) = P_n x^n + P_{n-1}x^{n-1} + \cdots + P_1 x + P_0.$$

Thus, we will have a solution if we can solve the following system of linear equations.

$$\begin{aligned}
cA_n &= P_n \\
bnA_n + cA_{n-1} &= P_{n-1} \\
an(n-1)A_n + b(n-1)A_{n-1} + cA_{n-2} &= P_{n-2} \\
a(n-1)(n-2)A_{n-1} + b(n-2)A_{n-2} + cA_{n-3} &= P_{n-3} \\
a(n-2)(n-3)A_{n-2} + b(n-3)A_{n-3} + cA_{n-4} &= P_{n-4} \\
a(n-3)(n-4)A_{n-3} + b(n-4)A_{n-4} + cA_{n-5} &= P_{n-5} \\
&\vdots \\
a6A_3 + b2A_2 + cA_1 &= P_1 \\
a2A_2 + bA_1 + cA_0 &= P_0
\end{aligned}$$

These are $n+1$ equations and $n+1$ unknowns. Since, $c \neq 0$ we can solve the first equation for A_n. Once we know A_n we can solve the second equation for A_{n-1}. And so on. $\square$

Corollary. If $c = 0$ we have that $y = 1$ is a solution of $ay'' + by' = 0$. Assume $b \neq 0$. Then to find a particular solution to $ay'' + by' = P(x)$ you can let $y_p(x) = A_n x^{n+1} + \cdots + A_1 x^2 + A_0 x$, plug this into the the differential equation and solve for the A_i's, or let $v = y_0$ and reduce the problem to a first order differential equation.

If $c = 0$ and $b = 0$, then $\{x, 1\}$ is a fundamental solution set of $ay'' = 0$. To find a particular solution to $ay'' = P(x)$ you can let $y_p(x) = A_n x^{n+2} + \cdots + A_1 x^3 + A_0 x^2$, plug this into the the differential equation and solve for the A_i's, or just integrate twice!

We leave the proofs to the reader.

Student Exercises 3. Find the general solution to each of the following.

a. $y'' - y' - 6y = x^3$
b. $y'' - 2y' = 3x^2 + 2e^{2x} + e^x$

1.3. **Forcing Terms in the form a polynomial times an exponential function.** Next we consider equations of the form $ay'' + by' + cy = P(x)e^{\alpha x}$. The prescription is basically the same.

Theorem 4. *Consider $ay'' + by' + cy = P(x)e^{\alpha x}$, where $P(x)$ is a polynomial of degree n. Let*

$$Q(x) = A_n x^x + \cdots A_1 x + A_0.$$

Let

$$y_p = \begin{cases} Q(x)e^{\alpha x} & \text{if } \alpha \text{ is not a root of } ar^2 + br + c, \\ xQ(x)e^{\alpha x} & \text{if } \alpha \text{ is a non-repeated root of } ar^2 + br + c, \\ x^2 Q(x)e^{\alpha x} & \text{if } \alpha \text{ is a repeated root of } ar^2 + br + c. \end{cases}$$

Then, one can find values for the A_i's such that y_p is a solution to the given differential equation.

Proof. Let $U(x) = x^s Q(x)$, where s can be 0, 1, or 2. Then $y_p' = U'e^{\alpha x} + \alpha U e^{\alpha x}$ and $y_p'' = U''e^{\alpha x} 2\alpha U' e^{\alpha x} + \alpha^2 e^{\alpha x}$. Plug these into the original differential equation to get

$$[aU'' + (2a\alpha + b)U' + (a\alpha^2 + b\alpha + c)U]e^{\alpha x} = P(x)e^{\alpha x}.$$

Divide both sides by $e^{\alpha x}$. For $s = 0$ we now have the same system of equation we had in Theorem 3. For $s = 1$ and $s = 2$ we use the same methods as in the corollary to Theorem 3. □

Example 7. Find the general solution to $y'' - y' - 6y = (x^2 + 1)e^{3x}$.

Solution. The general solution to the homogeneous problem $y'' - y' - 6y = 0$ is

$$y_h = C_1 e^{-2x} + C_2 e^{3x}.$$

Therefore, we let

$$y_p = x(Ax^2 + Bx + C)e^{3x}.$$

Thus,

$$y_p' = (3Ax^2 + 2Bx + C)e^{3x} + 3(Ax^3 + Bx^2 + Cx)e^{3x}$$

and

$$y_p'' = (6Ax + 2B)e^{3x} + 6(3Ax^2 + 2Bx + C)e^{3x} + 9(Ax^3 + Bx^2 + Cx)e^{3x}.$$

Thus,

$$y_p'' - y_p' - 6y_p = (6Ax + 2B)e^{3x} + 6(3Ax^2 + 2Bx + C)e^{3x} + 9(Ax^3 + Bx^2 + Cx)e^{3x} - ($$

We regroup the terms on the right hand side as follows

$$[(9A - 3A - 6A)x^3 + (18A + 9B - 3A - 3B - 6B)x^2 +$$

$$(6A + 12B + 9C - 2B - 3C - 6C)x + (2B + 6C - C)]e^{3x}.$$

This simplifies to

$$[15Ax^2 + (6A + 10B)x + (2B + 5C)]e^{3x}.$$

Setting this equal to $(x^2 + 1)e^{3x}$ gives three equations and three unknowns,

$$15A = 1, \quad 6A + 10B = 0, \quad 2B + 5C = 1.$$

Thus,

$$A = 1/15, B = -6/150 = -1/25, C = 27/125.$$

Therefore, the general solution is

$$y = C_1 e^{2x} + C_2 e^{3x} + \left(\frac{x^3}{15} - \frac{x^2}{25} + \frac{27x}{125} \right) e^{3x}.$$

□

Student Exercises 4.

a. $y'' - y' - 6y = xe^x$

b. $y'' - 2y' = x^2 e^{2x}$

1.4. **Forcing Terms involving sine or cosine.** Now we are going to up the ante once again. We shall consider equations of the forms

$$ay'' + by' + cy = P(x)e^{\alpha x} \sin \beta x$$

and

$$ay'' + by' + cy = P(x)e^{\alpha x} \cos \beta x$$

where $P(x)$ is a polynomial. We will do some simple examples before stating the general result.

Example 8. Find the general solution to $y'' - y' - 6y = \sin x$.

Solution. The solution to the corresponding homogeneous problem is

$$y_h = C_1 e^{-2x} + C_2 e^{3x}.$$

We guess that y_p is of the form $A \sin x + B \cos x$. Why is this a good guess? Because, $y_p'' - y_p' - 6y_p$ will be a linear combination of sines and cosines and then we might be able to choose A and B so that the cosines cancel out and the coefficients of sines add to one. Let's try.

$$y_p = A\sin x + B\cos x$$
$$y_p' = A\cos x - B\sin x$$
$$y_p'' = -A\sin x - B\cos x$$

Thus,

$$y_p'' - y_p' - 6y_p = (-A + B - 6A)\sin x + (-B - A - 6B)\cos x.$$

Thus, we require that $-7A+B = 1$ and $-A-7B = 0$. Therefore, $A = -7/50$ and $B = 1/50$. The general solution is

$$y = C_1e^{-2x} + C_2e^{3x} - \tfrac{7}{50}\sin x + \tfrac{1}{50}\cos x.$$

□

Example 9. Find the general solution to $y'' - y' - 6y = e^x \sin x$.

Solution. Only the details change. Again $y_h = C_1e^{-2x} + C_2e^{3x}$. Our guess for a particular solution this time is

$$y_p = Ae^x \sin x + Be^x \cos x.$$

Think about why this is a reasonable guess. Plug it in to the original equation and solve for A and B. You should get $A = -7/50$ and $B = -1/50$. Thus, the general solution is

$$y = C_1e^{-2x} + C_2e^{3x} - \tfrac{7}{50}e^x \sin x - \tfrac{1}{50}e^x \cos x.$$

□

One more example, then we will state the general result.

Example 10. Find the general solution to $y''-2y'+2y = e^x \sin x$.

Solution. Now, $y_h = C_1e^x \sin x + C_2e^x \cos x$. Thus, we cannot use $y_p = Ae^x \sin x + Be^x \cos x$, because the terms are solutions to the homogeneous problem. Let's try it, in case you are skeptical - as you should be.

$$y_p'' + 2y_p' + 2y_p = (2Be^x \sin x + 2Ae^x \cos x) -$$

$$2((A - B)e^x \sin x + (A + B)e^x \cos x) + 2(Ae^x \sin x + Be^x \cos x)$$

$$= (2B + 2(A - B) + 2A)e^x \sin x + (2A - 2(A + B) + 2B)e^x \cos x =$$

$$0e^x \sin x + 0e^x \cos x = 0 \neq e^x \sin x.$$

Thus, there are no values of A and B that would work. By now you should realize what needs to be done. We let

$$y_p = Axe^x \sin x + Bxe^x \cos x.$$

We leave it for you work out that the general solution is

$$y = C_1 e^x \sin x + C_2 e^x \cos x - \tfrac{x}{2} e^x \cos x.$$

□

Theorem 5. *Consider an equation of the form*

$$ay'' + by' + cy = P(x)e^{\alpha x} \sin \beta x$$

or

$$ay'' + by' + cy = P(x)e^{\alpha x} \cos \beta x,$$

where $P(x)$ is a polynomial of degree n. If $\alpha + \beta i$ is not a root of the characteristic polynomial, then let

$$y_p = A(x)e^{\alpha x} \sin \beta x + B(x)e^{\alpha x} \cos \beta x,$$

where

$$A(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0,$$

and

$$B(x) = B_n x^n + B_{n-1} x^{n-1} + \cdots + B_1 x + B_0.$$

Then there exists a solution for the coefficients, $A_n, \ldots, A_0, B_n, \ldots, B_0$, that gives a solution to the original equation.

If $\alpha + \beta i$ is a root of the characteristic polynomial, then let

$$y_p = xA(x)e^{\alpha x}\sin\beta x + xB(x)e^{\alpha x}\cos\beta x,$$

where $A(x)$ and $B(x)$ are as before. Then there exists a solution for the coefficients, $A_n, \ldots, A_0, B_n, \ldots, B_0$, that gives a solution to the original equation. As usual, the general solution is of the form $y = y_h + y_p$ where y_h is the general solution to the corresponding homogeneous problem

$$ay'' + by' + cy = 0.$$

Outline of Proof. In both cases you use substitution, compute the derivatives and simplify. You end up with $2n$ equations and $2n$ unknowns. Then you solve for the unknowns, that is the coefficients $A_n, \ldots, A_0$, and $B_n, \ldots, B_0$. □

These types of problems are too tedious to put on a test. Most people would do them by computer. However, you may be asked to just give y_h and y_p without solving for the coefficients.

Example 11. Find the general solution to $2y'' - 7y' + 6y = (x^2+1)e^{2x}\sin 3x$.

Solution. The characteristic polynomial is $2r^2 - 2r + 6 = (r-2)(2r-3)$. The roots are real and distinct. Thus,

$$y_h(x) = C_1e^{2x} + C_2e^{\frac{3x}{2}}.$$

Since $2+3i$ is not a root of the characteristic polynomial we have that y_p will take the following form.

$$y_p(x) = (A_2x^2 + A_1x + A_0)e^{2x}\sin 3x + (B_2x^2 + B_1x + B_0)e^{2x}\cos 3x.$$

Substitution into the given differential equation gives six equations in six unknowns. Solving for the coefficients gives

$$A_2 = -\tfrac{2}{37},\ \ A_1 = \tfrac{218}{12{,}321},\ \ A_0 = -\tfrac{1026}{50{,}653},$$

$$B_2 = -\frac{12{,}321}{1{,}367{,}631}, \quad B_1 = -\frac{95{,}904}{1{,}367{,}631}, \quad B_0 = \frac{3{,}449}{1{,}367{,}631}.$$

(If you think I did that by hand, you probably still believe Mexico is paying for "The Wall".) □

Example 12. Find the general solution to $y'' + y = x\cos x$.

Solution. The roots of the characteristic polynomial are $\pm i$. Hence,

$$y_h = C_1 \sin x + C_2 \cos x.$$

Notice we can write the forcing term as $xe^{0x}\cos 1x$. Since $0 + i$ is a root of the characteristic polynomial, the form of y_p is the following.

$$y_p = x(A_1x + A_0)\sin x + x(B_1x + B_0)\cos x.$$

Substitution gives four equations in four unknowns. Solving for these give

$$A_1 = B_0 = 1/4 \;\&\; A_0 = B_1 = 0.$$

Therefore the general solution is

$$y = C_1 \sin x + C_2 \cos x + \tfrac{x^2}{4}\sin x + \frac{x}{4}\cos x.$$

Note: When I did this on a computer using Maple I got the general solution to be

```
y(x) = sin(x)*_C2+ cos(x)*_C1+1/4*x*cos(x)+1/4*sin(x)*x∧2-1/4*sin(x).
```

The `-1/4*sin(x)` at the end does not seem to be right. But, notice it can be combined with `sin(x)*_C2`. It was just a quirk of the algorithm Maple used. □

Student Exercises 5.

a. $y'' - y' - 6y = e^x \sin x$
b. $y'' - 2y' + 2y = e^x \sin x$

1.5. **Summary.** Table 3.5.1 in your textbook is a summary. I have made an expanded form of it on the course website. For the problems below use it to find a suitable y_p.

...