Nonhomogeneous Linear Second Order Differential Equations

We will be looking at equations of the form

$$ay'' + by' + cy = g(x).$$

In 3.5 we develop the method of <u>Undetermined Coefficients</u>

Easy, but does not always apply.

In 3.6 we develop the method of <u>Variation of Parameters</u>

Harder, but always works.

$\boxed{3.5}$ Method of Undetermined Coefficients

<u>Ex</u> Solve $y' = xe^x$, that is find $y = \int xe^x\,dx$.

Assume you are too lazy to look up integration by parts, but you remember that the answer is in form $(Ax+B)e^x$. Then

$$y' = Ae^x + Axe^x + Be^x = xe^x + 0e^x$$

Thus $A=1$ and $B=-1$. We get,

$$\boxed{y = (x-1)e^x + C.}$$

Ex Solve $y' = x^3 e^x$, i.e. $y = \int x^3 e^x \, dx$.

It seems reasonable that the answer will be in the form

$$y = (Ax^3 + Bx^2 + Cx + D)e^x,$$

Then

$$y' = Ax^3e^x + 3Ax^2e^x + Bx^2e^x + 2Bxe^x + Cxe^x + Ce^x + De^x.$$

We need

$$Ax^3e^x + (3A+B)x^2e^x + (2B+C)xe^x + (C+D)e^x =$$

$$1x^3e^x + 0x^2e^x + 0xe^x + 0e^x.$$

That is

$$\begin{aligned} A &= 1 \\ 3A + B &= 0 \quad \Rightarrow B = -3 \\ 2B + C &= 0 \quad \Rightarrow C = 6 \\ C + D &= 0 \quad \Rightarrow D = -6 \end{aligned}$$

Then our solution is

$$y(x) = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

Ex Compute $y = \int x \cos x \, dx$.

Sol It seems reasonable that

$$y = Ax\cos x + Bx\sin x + C\cos x + D\sin x.$$

$$y' = -Ax\sin x + A\cos x + Bx\cos x + B\sin x$$
$$+ C\sin x + D\cos x$$

$$\cancel{-Ax\sin x + A\cos x}$$

$$Bx\cos x - Ax\sin x + (A+D)\cos x + (B-C)\sin x$$

$$= 1 \cdot x\cos x + 0\,x\sin x + 0\cos x + 0\sin x.$$

$$\begin{aligned} B &= 1 & & \\ A &= 0 & & A = 0 \\ A + D &= 0 & & D = 0 \\ B - C &= 0 & \Rightarrow & \; C = 1 \end{aligned}$$

Thus

$$\boxed{y = x\sin x + \cos x + C}$$

Now we go back to second order eq's.

<u>Ex</u> Consider $y'' - y' - 2y = 0$. (homogeneous)

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

$$y = C_1 e^{2x} + C_2 e^{-x}.$$

<u>Ex</u> What if $y'' - y' - 2y = 1$ (nonhomogeneous)

Notice $y = -\frac{1}{2}$ works.
Then any function of the form

$$y(x) = \underbrace{C_1 e^{2x} + C_2 e^{-x}}_{y_h} \underbrace{- \tfrac{1}{2}}_{y_p}$$

y_p = particular sol.

will work. y_h = homogeneous sol

<u>Ex</u> What if $y'' - y' - 2y = x$.

Try $y = -\frac{1}{2}x$. $0 - -\frac{1}{2} + x \neq x$.
Try $y = -\frac{1}{2}x + B$.

$$0 - \frac{-1}{2} + x - 2B = x$$

Then $B = \frac{1}{4}$.

Thus $y_p = -\frac{1}{2}x + \frac{1}{4}$

$$y = C_1 e^{2x} + C_2 e^{-x} - \frac{1}{2}x + \frac{1}{4}$$

Ex: $y'' + 2y' + y = 3x+1$

$(r+1)^2 \quad r = -1$

$y_h = c_1 e^{-x} + c_2 x e^{-x}$

Try $y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

$0 + 2A + Ax + B = 3x$

$A = 3$

$2A + B = 1$

$6 + B = 1 \Rightarrow B = -1$

$y = y_h + 3x - 1$

Ex $y'' + 2y' - 3y = x^2$ $\quad \boxed{y_h = c_1 e^{-3x} + c_2 e^{x}}$

Try $y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$2A + \underline{4Ax} + 2B - \underline{3Ax^2} - \underline{3Bx} - 3C = x^2$

$-3A = 1$

$A = -\frac{1}{3}$

$(4A - 3B)\cancel{x} = 0 \quad -\frac{4}{3} = 3B \quad B = -\frac{4}{9}$

$2B + 2A - 3C = 0$

$-\frac{8}{9} - \frac{2}{3} = 3C \quad C = \frac{1}{3}\,\frac{-14}{9} = \frac{-14}{27}$

$\boxed{y_p = -\frac{1}{3}x^2 - \frac{4}{9}x - \frac{14}{27} \qquad y = y_h + y_p.}$

Ex $$y'' + 2y' - 3y = 2\sin 3x.$$

$$y_h = C_1 e^{-3x} + C_2 e^{x}.$$

Let $y = y_p = A\sin 3x + B\cos 3x$.

$$y' = 3A\cos 3x - 3B\sin 3x$$

$$y'' = -9A\sin 3x - 9B\cos 3x.$$

plug in:

$$(-9A\sin 3x - 9Bc) + 2(3Ac - 3Bs) - 3(As + Bc)$$

$$(-9A - 6B - 3A)s + (-9B + 6A - 3B)c = \underline{2\sin 3x}.$$

$$-12A - 6B = 2 \qquad -6A - 3B = 1$$

$$6A - 12B = 0 \qquad \underline{6A - 12B = 0}$$

$$-15B = 1$$

$$B = -1/15$$

$$A = 2B = -2/15.$$

$$y_p = -\tfrac{2}{15}\sin(3x) - \tfrac{1}{15}\cos(3x)$$

$y = y_h + y_p$ $\Rightarrow$ general solution.

Ex $\quad y'' + y = \sin x \qquad y_h = C_1 \sin x + C_2 \cos x$

Try $y_p = Ax\sin x + Bx\cos x$

You can show that $A = 0$, $B = -\frac{1}{2}$.

Ex $\quad y'' + y = e^x$

$$y_p = Ae^x$$
$$y_p'' = Ae^x$$

$$Ae^x + Ae^x = e^x$$
$$A = \frac{1}{2}$$

$$y = C_1 \sin x + C_2 \cos x + \frac{1}{2}e^x$$

Ex $\quad y'' + 2y + y = e^{-x} \qquad y_h = C_1 e^{-x} + C_2 x e^{-x}$

$$y_p = Ax^2 e^{-x}$$

You can show that $A = \frac{1}{2}$.

Ex $y'' + 2y' - 3y = 2e^{3x}$.

Sol Try $y_p = Ae^{3x}$

$$9Ae^{3x} + 6Ae^{3x} - 3Ae^{3x} = 2e^{3x}$$

$$9A + 6A - 3A = 2$$

$$12A = 2$$

$$A = 1/6.$$

$$y_p = \frac{1}{6} e^{3x}.$$

General solution: $y = C_1 e^{-3x} + C_2 e^{x} + \frac{1}{6} e^{3x}$.

Ex $y'' + 2y' - 3y = 5e^x$

Now $y_p = Ae^x$ won't work since we know $y_p'' + 2y_p' - 3y_p = 0$.

Try $y_p = Axe^x$.

$$y_p' = Ae^x + Axe^x$$
$$y_p'' = 2Ae^x + Axe^x$$

plug in.

$$2Ae^x + Axe^x + 2Ae^x + 2Axe^x - 3Axe^x$$
$$= 0xe^x + 5e^x.$$

$$\underbrace{(A + 2A - 3A)}_{=0}xe^x + (2A + 2A)e^x = 0xe^x + 5e^x$$

$$4A = 5$$
$$A = 5/4.$$

$$y_p = \frac{5}{4}xe^x$$

$$\boxed{y = C_1e^{3x} + C_2e^x + \frac{5}{4}xe^x.}$$

Discuss Table 3.5.1