Chapter 4: In brief

Ch 4 is exactly like Ch 3. We will just do a few examples.

Ex 1

$$y'''' - 9y'' + 4y' + 2y = 0$$

$$r^4 - 9r^2 + 4r + 2 = 0$$
$$(r-2)^2(r+3)(r+1) = 0$$
$$r = 2, 2, -3, -1$$

General solution is
$$y = C_1 e^{2x} + C_2 x e^{2x} + C_3 e^{-3x} + C_4 e^{-x}.$$

If we had initial conditions of the form
$$y(0) = a,\ y'(0) = b,\ y''(0) = c,\ y'''(0) = d$$
we could find values for C_1, C_2, C_3 and C_4.

Ex 2

$$y''' + 3y'' + 3y' + y = 0$$

$$r^3 + 3r^2 + 3r + 1 = 0$$
$$(r+1)^3 = 0$$
$$r = -1, -1, -1.$$

$$y = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}.$$

Ex 3 $\quad y'''' + 2y'' + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0 \quad r = \pm i, \pm i$

$y = C_1 \cos t + C_2 \sin t + C_3 t\cos t + C_4 t\sin t$

Ex 4 $\quad y'''' + 2y'' + y = \sin t + t^2$

y_h is same as Ex 3. For y_p use

$At^2\sin t + Bt^2\cos t + Ct^2 + Dt + E$.

You can do them separately.

Let $y_p = Ct^2 + Dt + E$.

$y_p' = 2Ct + D$

$y_p'' = 2C$

$y_p''' = 0$

Thus $2(2C) + Ct^2 + Dt + E = t^2$

Thus $C = 1$, $D = 0$ $E = -4C = -4$.

This $y_{p_1} = t^2 - 4$.

Let $y_p = At^2 \sin t + Bt^2 \cos t$. This will get messy.

$$y_p' = 2Ats + At^2c + 2Btc - Bt^2s$$

$$y_p'' = 2As + 2Atc + 2Atc - At^2s + 2Bc - 2Bts - 2Bts - Bt^2c$$

$$= 2As + 4Atc - At^2s + 2Bc - 4Bts - Bt^2c$$

$$y_p''' = 2Ac + 4Ac - 4Ats - 2Ats - At^2c - 2Bs - 4Bs - 4Btc - 2Btc + Bt^2s$$

$$= 6Ac - 6Ats - At^2c - 6Bs - 6Btc + Bt^2s$$

$$y_p'''' = -6As - 6As - 6Atc - 2Atc + At^2s - 6Bc - 6Bc + 6Bts + 2Bts + Bt^2c$$

$$= -12As - 8Atc + At^2s - 12Bc + 8Bts + Bt^2c$$

$$y_p'''' + 2y_p'' + y_p =$$

no!

$$(A - 2A + A)t^2s + (B - \cancel{A}2B + B)t^2c$$

$$(8B - 2\cdot 4B)ts + (-8A + 2\cdot 4A)tc$$

$$(-12A + 4A)s + (-12B + 2\cdot 2B)c$$

$$-8A = 1 \Rightarrow A = -\tfrac{1}{8} \qquad -8B = 0 \Rightarrow B = 0$$

$$\boxed{\text{Thus } y_{p_2} = -\tfrac{1}{8}t^2 \sin t.}$$

putting it all together, the general solution is

$$y = y_h + y_{p_1} + y_{p_2} =$$

$$\boxed{C_1 \cos t + C_2 \sin t + C_3 t \cos t + C_4 t \sin t + t^2 - 4 - \frac{1}{8} t^2 \sin t}$$

Next I'll do one example for variation of parameters, but it would help to read over 4.4 a bit.

Then I'll do an interesting applied example of a coupled mass-spring system.

Example of 3rd Order Eq. using Variatoon of Parmeters.

It is very similar to the 2nd order case but the Wronskians are more complicated. There are "Sub-Wronskians" that are needed. Let y_1, y_2, y_3 we twice differentiable functions. Then

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Then the subWronskians, called W_i, $i=1,2,3$, are formed by replacing the i^{th} column with $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus,

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} = y_2 y_3' - y_3 y_2'.$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = - \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix} = y_3 y_1' - y_1 y_3'$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Or, $W_1 = W(y_2\ y_3)$, $W_2 = -W(y_1, y_3)$, $W_3 = W(y_1, y_2)$

Example 4.3 #1 Find general solution to $y''' + y' = \tan t$ using variation of parameters.

Solution $r^3 + r = r(r^2+1) = 0 \Rightarrow r = 0, \pm 1.$

Thus $y_h = C_1 + C_2 \cos t + C_3 \sin t$. $y_1 = 1, y_2 = \cos t, y_3 = \sin t$.

$W(1,c,s) = 1 \quad W_1 = 1 \quad W_2 = -c \quad W_3 = -s.$

Then

$u_1 = \int \frac{\tan t \cdot W_1}{W} dt = \int \tan t \, dt = \int \frac{\sin t}{\cos t} dt = -\ln|\cos t| = \ln(\sec t)$

$u = \cos t \quad du = -\sin t$

$u_2 = \int \frac{\tan t \, W_2 \, dt}{W} = -\int \sin t \, dt = \cos t$

$u_3 = \int \frac{\tan t \, W_3}{W} dt = -\int \frac{s^2}{c} dt = \int \frac{c^2-1}{c} dt = \int \cos t - \sec t \, dt$

$= \sin t - \ln|\sec t + \tan t|$

Now $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 =$

$\ln(\sec t) + \cos^2 t + \sin^2 t - \sin \ln|s+t| =$

$\ln(\sec t) - \sin t \ln|s+t| + 1$

can ignore

We plug in and check just to be sure.

$$y_p' = \tan t - \cos t \ln|s+t| - \sin t \,\frac{st+s^2}{s+t} = \tan t$$

$$= -\cos t \ln|s+t|$$

$$y_p'' = \sin t \ln|s+t| - \cos t \,\frac{st+s^2}{s+t} = 1$$

$$y_p''' = \cos t \ln|s+t| + \sin t \,\frac{st+t^2}{s+t} = \tan t$$

Thus $y_p''' + y_p' =$

$$\cos t \ln|s+t| + \tan t - \cos t \ln|s+t| = \tan t \checkmark$$

General solution is

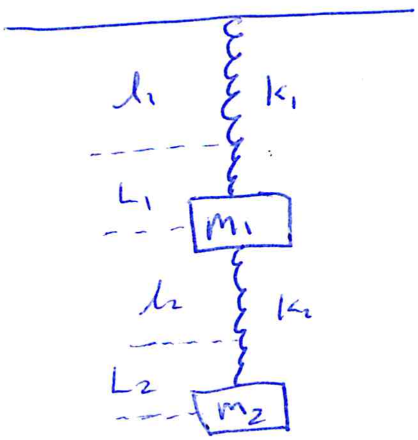
$$\boxed{y(t) = C_1 + C_2 \cos t + C_3 \sin t + \ln(\sec t) - \sin t \ln(\sec t + \tan t)}$$

Example (4.2 #39)

We consider a coupled mass-spring system as shown below. We assume no damping, $\gamma=0$.

We are given $m_1 = m_2 = 1$, $k_1 = 3$, $k_2 = 2$.

Equilibrium

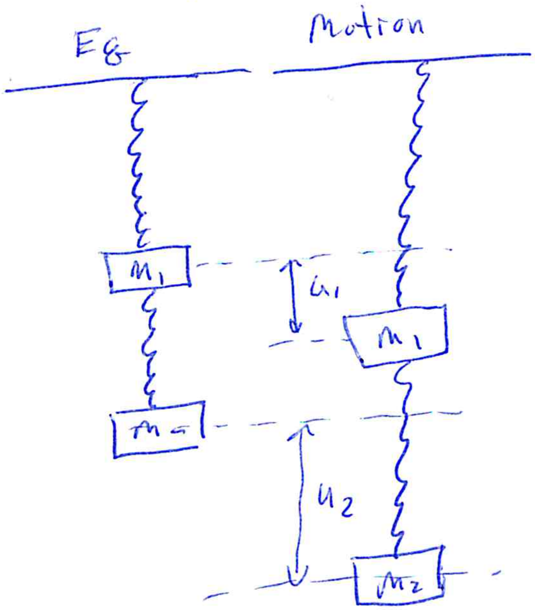


We analyze the forces on m_1 and m_2 during equilibrium.

$$F_1 = m_1 g - k_1 L_1 + k_2 L_2 = 0$$

$$F_2 = m_2 g - k_2 L_2 = 0$$

Now set it into motion. Let $u_1(t)$ and $u_2(t)$ be the respective displacements from equilibrium for m_1 and m_2.



$$F_1 = m_1 g - k_1(L_1+u_1) + k_2(L_2+u_2-u_1)$$

$$= \underbrace{m_1 g - k_1 L_1 + k_2 L_2}_{=0} - (k_1+k_2)u_1 + k_2 u_2$$

Thus $m_1 u_1'' = -(k_1+k_2)u_1 + k_2 u_2$.

We write this as

$$\boxed{m_1 u_1'' + (k_1+k_2)u_1 = k_2 u_2}$$

$$F_2 = m_2 g - k_2(L_2 + u_2 - u_1)$$

$$= \underbrace{m_2 g - k_2 L_2}_{=0} - k_2 u_2 + k_1 u_1$$

Thus, $\boxed{m_2 u_2'' + k_2 u_2 = k_2 u_1}$

Plugging in $m_1 = m_2 = 1, k_1 = 3, k_2 = 2$ we get

$$\boxed{\begin{aligned} u_1'' + 5u_1 &= 2u_2 \quad (i) \\ u_2'' + 2u_2 &= 2u_1 \quad (ii) \end{aligned}}$$

We convert these into a 4^{th} order diff eq as follows.

By (i) $u_2 = \frac{u_1'' + 5u_1}{2}$. Plug into (ii)

$$\left(\frac{u_1''+5u_1}{2}\right)'' + 2\left(\frac{u_1''+5u_1}{2}\right) = 2u_1 \quad \text{or}$$

$$\frac{1}{2}u_1'''' + \frac{5}{2}u_1'' + u_1'' + 5u_1 = 2u_1 \quad \text{or}$$

$$\boxed{u_1'''' + 7u_1'' + 6u_1 = 0.}$$

We next find the general solution for u_1

$$r^4 + 7r^2 + 6 = 0$$

$$(r^2+6)(r^2+1) = 0$$

$$r = \pm i\sqrt{6}, \pm i.$$

Thus

$$u_1(t) = C_1 \cos t + C_2 \sin t + C_3 \cos\sqrt{6}t + C_4 \sin\sqrt{6}t$$

And,

$$u_2(t) = \frac{u_1''+5u_1}{2} = 2C_1 \cos t + 2C_2 \sin t - \frac{1}{2}C_3 \cos\sqrt{6}t + \frac{1}{2}C_4 \sin\sqrt{6}t$$

(as you will check)

Given init conditions we can solve for C_1, C_2, C_3, C_4.

Ex Suppose $u_1(0)=0$, $u_1'(0)=0$, $u_2(0)=7$, $u_2'(0)=0$.

Then

$u_1(0)=0 \Rightarrow \quad C_1 + C_3 = 0$

$u_1'(0)=0 \Rightarrow \quad C_2 + C_4\sqrt{6} = 0$

$u_2(0)=7 \Rightarrow \quad 2C_1 - \frac{1}{2}C_3 = 7$

$u_2'(0)=0 \Rightarrow \quad 2C_2 + \frac{1}{2}\sqrt{6}\,C_4 = 0$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \sqrt{6} \\ 2 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 0 & \frac{1}{2}\sqrt{6} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

~~Then $C_1 = 0$ ... $C_2 = 0$, $C_4 =$ ... $C_3 =$~~

$C_1 = \frac{14}{5} \quad C_3 = -\frac{14}{5}$

$C_2 = C_4 = 0.$

plots are on the next page.

```
> plot(14/5*(cos(t) - cos(sqrt(6)*t)),t=0..30,thickness=2,color=black);
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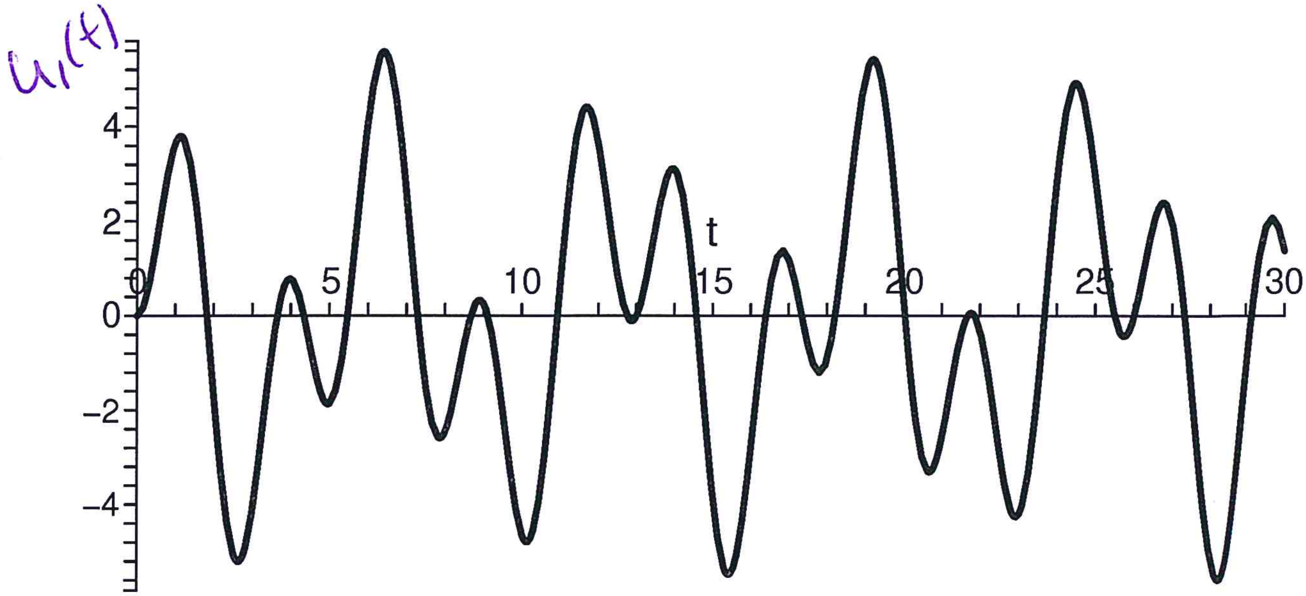


```
> plot(28/5*cos(t) - 7/5*cos(sqrt(6)*t), t=0..30,thickness=2,color=black);
```