Lecture Notes for Ch 5
Series Solutions
Introduction and Overview

You are responsible for knowing the material in Sections 1 - 3 of Chapter 5. Section 5.1 is a review of infinite series from *Calculus II*. Read this on your own. You should know how to do problems 1-27 in Section 5.1. Do not turn these in. You may be quizzed on them.

I will cover 5.2 and 5.3 in reverse order. This is because 5.3 is easier. The basic idea is that if we cannot find the exact solution to a differential equation perhaps we can find the Taylor series of the solution and use some n-order Taylor polynomial as an approximation to the solution.

In 5.3 we just compute a few terms of the series solution. In 5.2 we show how to find a recursive formula for the coefficients of the series solution in some cases.

This material is extremely tedious. Be patient. Read slowly. These notes mostly consist of examples. After reading through an example, go back and work out all the details on a separate sheet of paper. Compare the examples with the homework problems and pause to work on them as you go through these notes. Good luck, you'll need it!

Lecture Notes for Ch 5
Series Solutions

Given a differential equation, we suppose the solution has a Taylor series. Recall that if $y(t)$ has a Taylor series centered about $t = 0$ the formula for it is

$$y(t) = y(0) + y'(0)t + \frac{y''(0)}{2!}t^2 + \frac{y'''(0)}{3!}t^3 + \cdots = \sum_{n=0}^{\infty} \frac{y^{[n]}(0)t^n}{n!}.$$

More generally, a Taylor series could be centered about any number, say $t = c$. Then

$$y(t) = \sum_{n=0}^{\infty} \frac{y^{[n]}(c)(t-c)^n}{n!}.$$

If we write $y(t) = \sum_{n=0}^{\infty} a_n(t-c)^n$ then of course $a_n = \dfrac{y^{[n]}(c)}{n!}$.

The idea is to plug the series for $y(t)$ into the differential equation and deduce the a_n's. Think of it as the Method of Undermined Coefficients on steroids. We start with a very naive example, a first order equation that you already know how to solve.

Example 1. Solve $y' = y$ with $y(0) = 2$ using a power series for y centered at zero.

Solution. Let $y = a_0 + a_1 t + a_2 t^2 + \cdots = \sum_{n=0}^{\infty} a_n t^n$. Then, since $y(0) = a_0$ we know $a_0 = 2$.
Next we compute the first derivative,

$$y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots = \sum_{n=0}^{\infty} n a_n t^{n-1}.$$

Thus, $y'(0) = a_1$. But $y'(0) = y(0) = 2$. Hence $a_1 = 2$.

Next we compute the second derivative,

$$y''(t) = 2a_2 + 3 \cdot 2a_3 t + 4 \cdot 3a_4 t^2 + \cdots = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}.$$

Thus, $y''(0) = 2a_2$. But $y'' = (y')' = y' = y$. Thus $y''(0) = y(0) = 2$. Hence $a_2 = 1$.

And then the third. $y'''(0) = 3 \cdot 2a_3 = 3!a_3$. But $y'''(0) = y''(0) = 2$. Hence $a_3 = 2/3!$.

Finally, we find $y''''(0) = 4!a_4$. But $y''''(0) = 2$. Hence $a_4 = 2/4!$.

If you do a few more terms, you should see that the pattern is $a_n = 2/n!$. Thus,

$$y(t) = \sum_{n=0}^{\infty} \frac{2}{n!} t^n.$$

But this is just the Taylor series of $2e^t$.

Series Solutions of
Second Order Linear Differential Equations

We will do an example with a second order differential equation with constant coefficients. This is not the best way to do such a problem, but we want to illustrate the series method with easy examples to start.

Example 2. Find the first five terms of the series solution of $y'' + 2y' + y = 0$ with $y(0) = 1$, $y'(0) = 2$.

Solution. Since the initial condition is at zero our series will be centered at zero as well. Suppose $y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = \sum_{n=0}^{\infty} a_n t^n$. Then $a_n = \frac{y^{[n]}(0)}{n!}$.

Now, $y(0) = a_0$ implies $a_0 = 1$. Likewise $y'(0) = a_1$ implies $a_1 = 2$. It is true in general that the initial conditions determine a_0 and a_1.

Next, notice that $y''(t) = -2y'(t) - y(t)$. Thus, $y''(0) = -2y'(0) - y(0) = -2 \cdot 2 - 1 = -5$. Therefore,

$$a_2 = \frac{y''(0)}{2!} = -\frac{5}{2}.$$

Now, we need to compute $y'''(t)$ and evaluate it at $t = 0$.

$$y'''(t) = (y''(t))' = (-2y'(t) - y(t))' = -2y''(t) - y'(t).$$

Thus,

$$y'''(0) = -2y''(0) - y'(0) = -2 \cdot (-5) - 2 = 8.$$

Therefore,

$$a_3 = \frac{y'''(0)}{3!} = \frac{8}{6} = \frac{4}{3}.$$

One more! We need to compute $y''''(t)$ and evaluate it at $t = 0$.

$$y''''(t) = (-2y''(t) - y'(t))' = -2y'''(t) - y''(t).$$

Therefore.

$$y''''(0) = -2y'''(0) - y''(0) = -2 \cdot 8 - (-5) = -11.$$

Thus,

$$a_4 = \frac{y''''(0)}{4!} = -\frac{11}{24}.$$

Finally, we put this together to get,

$$y(t) \approx 1 + 2t - \frac{5}{2}t^2 + \frac{4}{3}t^3 - \frac{11}{24}t^4.$$

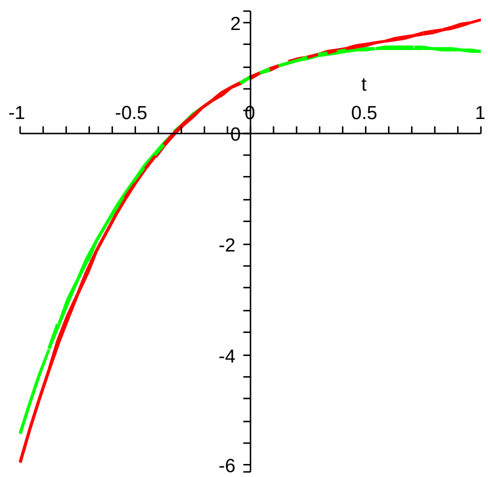
In this example, we did not find a general formula for a_n. If we pushed it further we might be able to, but this is good enough for now.

Examining the Last Example

Recall we wanted to solve $y'' + 2y' + y = 0$ with $y(0) = 1$ and $y'(0) = 2$. It is easy enough to find the exact solution. It is

$$y = e^{-t} + 3te^{-t}.$$

Below we plot this along with the forth degree Taylor polynomial we found, $y = 1 + 2t - \frac{5}{2}t^2 + \frac{4}{3}t^3 - \frac{11}{24}t^4$. The green curve is the exact solution and the red curve is the approximation.



Example

Example 3. Find the first five terms of the series solution of $y'' + (\sin x)y' + (\cos x)y = 0$, with $y(0) = 0$ and $y'(0) = 1$.

Solution. Let $y = \sum_{n=0}^{\infty} a_n x^n$ with $a_n = \frac{y^{[n]}(0)}{n!}$. Then we have the following calculations.

$$y(0) = 0 \implies a_0 = 0 \quad \& \quad y'(0) = 1 \implies a_1 = 1.$$

Next we solve for y'', evaluate it at $x = 0$, and then compute a_2.

$$y'' = -(\sin x)y' - (\cos x)y.$$

Thus,

$$y''(0) = -0 \cdot 1 - 1 \cdot 0 = 0 \implies a_2 = 0/2! = 0.$$

Next, we find y''', evaluate it at $x = 0$, and then compute a_3.

$$\begin{aligned} y''' &= (-(\sin x)y' - (\cos x)y)' \\ &= -\left((\cos x)y' + (\sin x)y''\right) - (-(\sin x)y + (\cos x)y') \\ &= -(\sin x)y'' - 2(\cos x)y' + (\sin x)y. \end{aligned}$$

Thus,

$$y'''(0) = -0 \cdot 0 - 2 \cdot 1 \cdot 1 + 0 \cdot 0 = -2 \implies a_3 = -2/3! = -1/3.$$

Finally, we find y'''', evaluate it at $x = 0$, and then compute a_4.

$$\begin{aligned} y'''' &= (-(\sin x)y'' - 2(\cos x)y' + (\sin x)y)' \\ &= -(\cos x)y'' - (\sin x)y''' + 2(\sin x)y' - 2(\cos x)y'' + (\cos x)y + (\sin x)y' \\ &= -(\sin x)y''' - 3(\cos x)y'' - 3(\sin x)y' - (\cos x)y. \end{aligned}$$

Thus,

$$y''''(0) = -0(-2) - 3 \cdot 1 \cdot 0 - 3 \cdot 0 \cdot 1 - 1 \cdot 0 = 0 \implies a_4 = 0/4! = 0.$$

Putting this all together we get

$$y(x) \approx x - \frac{1}{3}x^3.$$

Examining the Previous Example

Recall we are studying $y'' + (\sin x)y' + (\cos x)y = 0$, with $y(0) = 0$ and $y'(0) = 1$.

We do not have the methods to find the exact solution. Maple gives the following for the general solution

$$y(x) = e^{\cos x}\left(C_1 \int_0^x e^{-\cos s}\, ds + C_2\right).$$

The answer is given in terms of an integral that cannot be done in closed form. We will use $y(0) = 0$ and $y'(0) = 1$ to find C_1 and C_2. We will then check that it works by plugging it into the original differential equation.

$$y(0) = e(0 + C_2) = 0 \quad \Longrightarrow \; C_2 = 0.$$

$$y'(x) = -\sin x\, e^{\cos x} C_1 \int_0^x e^{-\cos s}\, ds + C_1 e^{\cos x} e^{-\cos x} = -\sin x\, e^{\cos x} C_1 \int_0^x e^{-\cos s}\, ds + C_1.$$

Hence,

$$C_1 = y'(0) = 1.$$

Now,

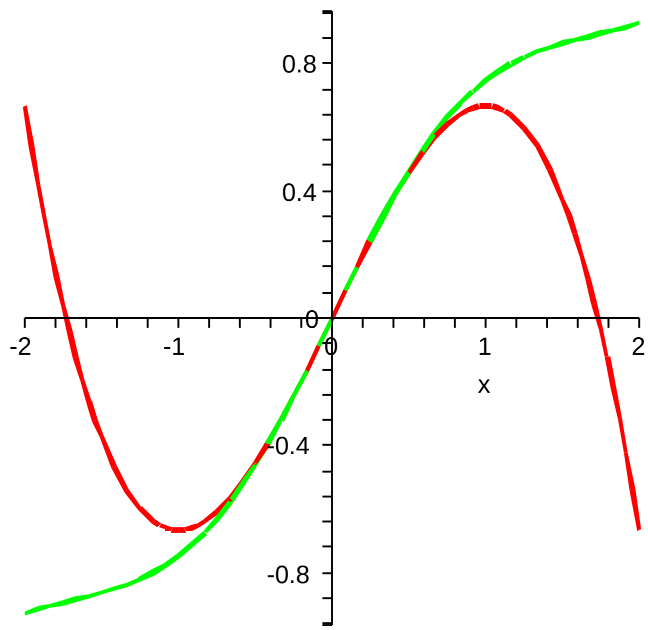
$$y''(x) = -\cos x\, e^{\cos x} \int_0^x e^{-\cos s}\, ds + \sin^2 x\, e^{\cos x} \int_0^x e^{-\cos s}\, ds - \sin x.$$

Thus,

$$\begin{aligned} y'' + (\sin x)y' + (\cos x)y &= -\cos x\, e^{\cos x} \int_0^x e^{-\cos s}\, ds + \sin^2 x\, e^{\cos x} \int_0^x e^{-\cos s}\, ds - \sin x \\ &+ \sin x \left(-\sin x\, e^{\cos x} \int_0^x e^{-\cos s}\, ds + 1\right) + \cos x\, e^{\cos x} \int_0^x e^{-\cos s}\, ds \\ &= \left(-\cos x + \sin^2 x - \sin^2 + \cos x\right) e^{\cos x} \int_0^x e^{-\cos s}\, ds. \\ &= 0. \checkmark \end{aligned}$$

Finally, we plot the exact solution along with the approximation. The green curve is the exact solution and the red curve is the approximation.

1in



Example not centered at zero

Example 4. Find the first five terms of the series solution of $y'' + xy' + y = 0$ with $y(2) = 3$ and $y'(2) = 1$.

Solution. Now the series will not be centered at zero but at $x_0 = 2$. Let $y = a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + \cdots = \sum_{n=0} a_n(x-2)^n$ with $a_n = \dfrac{y^{[n]}(2)}{n!}$. Then we have the following calculations.

$$y(2) = 3 \implies a_0 = 3 \quad \& \quad y'(2) = 1 \implies a_1 = 1.$$

Next we solve for y'', evaluate it at $x = 2$, and then compute a_2.

$$y'' = -xy' - y$$

Thus,

$$y''(2) = -2 \cdot 1 - 3 = -5 \implies a_2 = -5/2! = -5/2.$$

Next, we find y''', evaluate it at $x = 2$, and then compute a_3.

$$y''' = (-xy' - y)' = -y' - xy' - y' = -xy'' - 2y'.$$

Thus,

$$y'''(2) = -2(-5) - 2 \cdot 1 = 8 \implies a_3 = 8/3! = 4/3.$$

Finally, we find y'''', evaluate it at $x = 2$, and then compute a_3.

$$y'''' = (-xy'' - 2y')' = -y'' - xy''' - 2y'' = -xy''' - 3y''.$$

Thus,

$$y''''(2) = -2 \cdot 8 - 3(-5) = -1 \implies a_4 = -1/4! = -1/24.$$

Putting this all together we get

$$y(x) \approx 3 + (x-2) - \frac{5}{2}(x-2)^2 + \frac{4}{3}(x-2)^3 - \frac{1}{24}(x-2)^4.$$

When does this method work?

Failed Example 1. Can we find a power series $y = \sum_{n=0}^{\infty} a_n x^n$ that solves $y'' - \sqrt{x}\, y = 0$ with $y(0) = 1$ and $y'(0) = 2$? Let's try. Clearly, $a_0 = 1$ and $a_1 = 2$. Next

$$y'' = \sqrt{x}\, y \quad \Longrightarrow \quad y''(0) = \sqrt{0}\, y(0) = 0 \quad \Longrightarrow \quad a_2 = 0/2! = 0.$$

So far, so good. Now

$$y''' = (y'')' = (\sqrt{x}\, y)' = \frac{1}{2\sqrt{x}} y + \sqrt{x} y'.$$

But now

$$y'''(0) = \frac{1}{2\sqrt{0}} y(0) + \sqrt{0} y'(0),$$

which is undefined! We conclude that there is no series solution centered about zero.

Failed Example 2. Can we find a power series $y = \sum_{n=0}^{\infty} a_n (x-1)^n$ that solves $(x-1)y'' - xy' - 2y = 0$ with $y(1) = 2$ and $y'(1) = 3$? Let's try. Clearly, $a_0 = 2$ and $a_1 = 3$. Next

$$y'' = \frac{xy' + 2y}{x - 1}.$$

But then

$$y''(1) = \frac{1 \cdot 3 + 2 \cdot 2}{1 - 1},$$

which is undefined.

What went wrong?

A Theorem on Power Series Solutions

Theorem. Consider

$$y'' + p(x)y' + q(x)y = 0.$$

If $p(x)$ and $q(x)$ have Taylor series centered about $x = c$ then we say c is an **ordinary point** of the given differential equation. Otherwise, c is a **singular point**. In the last two "failed examples" we tried to use a series centered on a singular point. This is not good. (See Sections 5.4-5.8 for more on this.) But, if c is an ordinary point it is guaranteed that the solution exists and has a power series centered at c.

What is the radius of convergence?

Suppose $y(t) = \sum_{n=0}^{\infty} a_n (t-c)^n$ is a series solution to

$$y'' + p(t)y' + q(t)y = 0.$$

Suppose the Taylor series centered at c of $p(t)$ and $q(t)$ exist and have radii of convergence R_p and R_q, respectively. Then if R_y is the radius of convergence of series centered at c for $y(t)$ we have

$$R_y \geq \min\{R_p, R_q\}.$$

From *Calculus II* you have the tools to find R_p and R_q. Here is a trick that you probably did not cover that is useful for rational functions. Recall a rational function is the ratio of two polynomials. Let

$$r(x) = \frac{f(x)}{g(x)}$$

be a rational function where $f(x)$ and $g(x)$ are polynomials with no common factors. Let $x_1, x_2, \ldots x_k$ be all the zeros pf $g(x)$ in the **complex plane**. If c is any real or complex number not equal to any of the zeros of $g(x)$, then $r(x)$ will have a Taylor series centered at c with radius of convergence given by

$$R = \min\{|x_1 - c|, |x_2 - c|, \ldots |x_k - c|\}.$$

We give some examples on the next page.

Radius of Convergence Examples

(1) The radius of convergence for the Taylor series centered at $c = 0$ for $\dfrac{1}{1-x}$ is $R = 1$.

(2) The radius of convergence for the Taylor series centered at $c = 10$ for $\dfrac{1}{1-x}$ is $R = 9$.

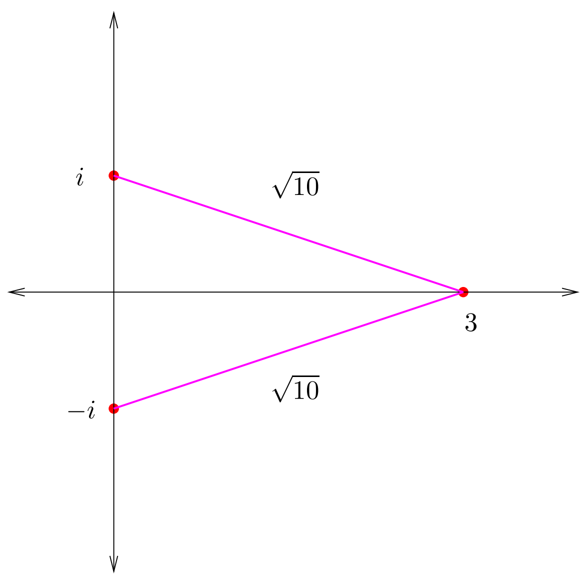
(3) The radius of convergence for the Taylor series centered at $c = 10$ for $\dfrac{x^3+7x}{1-x}$ is $R = 9$. (As long as 1 is not a root of the numerator, it makes no difference in the value of R.)

(4) The radius of convergence for the Taylor series centered at $c = 0$ for $\dfrac{1}{1-x^2}$ is $R = 1$.

(5) The radius of convergence for the Taylor series centered at $c = -0.7$ for $\dfrac{1}{1-x^2}$ is $R = 0.3$.

(6) The radius of convergence for the Taylor series centered at $c = 0$ for $\dfrac{1}{1+x^2}$ is $R = 1$, since the denominator has zeros at $\pm i$.

(7) The radius of convergence for the Taylor series centered at $c = 3$ for $\dfrac{1}{1+x^2}$ is $R = \sqrt{10}$. See figure below.



Radius of Convergence Examples

(1) Consider $y'' + (\sin x)y' + (\cos x)y = 0$. Then for any c the radius of convergence of the series solution will be infinite since the Taylor series for $\sin x$ and $\cos x$ have infinite radii of convergence.

(2) Consider $y'' + \dfrac{x}{1+x^2}y' + \dfrac{1}{x+2} = 0$.

If $c = 0$, then $R \geq 1$.

If $c = -5$, then $R \geq 3$.

If $c = 3$, then $R \geq \sqrt{10}$.

If $c = -1/2$, then $R \geq \sqrt{5}/2$.

(3) Consider $(x-1)y'' + \dfrac{x^3}{2-x}y' + \dfrac{1}{x-3}y = 0$. Remember you have to divide through by the $x - 1$. Thus the zeros are 1,2, and 3.

If $c = 0$, then $R \geq 1$.

If $c = 2.2$, then $R \geq 0.2$.

if $c = 23$, then $c \geq 20$.

Recursive Formulas

For a sequence of numbers $(a_n)_{n=0}^{\infty}$ it is ideal if we can find a formula for a_n as a function of n. For example, the sequence $(0, 1, 4, 9, 16, 25, 36, 49, \dots)$ is given by $a_n = n^2$. But, sometimes this is difficult or impossible to do. Consider the *Fibonacci sequence*,

$$(f_n)_{n=0}^{\infty} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots).$$

It is defined as follows, start with 0 and 1 as the first two terms, then each term after that is the sum of the two terms before it. That is

$$f_0 = 0, f_1 = 1, \& f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

This is called a **recursive formula**. For second order linear differential equations finding a recursive formula for the terms of the power series if often the best we can do.

Doing the Index Shift

No, it is not the latest dance craze. It is just a handy trick when working with summation notation. If you want add one or more power series together and express the result as a single power series, you too will find yourself doing the index shift.

Example. Suppose we want to add $\displaystyle\sum_{n=0}^{\infty} a_n t^n$ and $\displaystyle\sum_{n=0}^{\infty} b_n t^{n+2}$. We would rewrite the second sum as $\displaystyle\sum_{n=2}^{\infty} b_{n-2} t^n$. Then we'd get

$$\sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} b_n t^{n+2} = a_0 + a_1 t + \sum_{n=2}^{\infty} a_n t^n + \sum_{n=2}^{\infty} b_{n-2} t^n = a_0 + a_1 t + \sum_{n=2}^{\infty} (a_n + b_{n-2}) t^n.$$

Notice we had to treat the first two terms of the first sum separately. If you are having trouble following the details, write out the terms of the sums until you see what is happening.

Example. Let's do another one.

$$\begin{aligned}\sum_{n=0}^{\infty} n^2x^n + \sum_{n=0}^{\infty}(n+1)x^{n+1} + \sum_{n=2}^{\infty} a_n x^{n-2} &= \sum_{n=0}^{\infty} n^2x^n + \sum_{n=1}^{\infty} nx^n + \sum_{n=0}^{\infty} a_{n+2}x^n \\ &= \left(0 + \sum_{n=1}^{\infty} n^2x^n\right) + \sum_{n=1}^{\infty} nx^n + \left(a_2 + \sum_{n=1}^{\infty} a_{n+2}x^n\right) \\ &= a_2 + \sum_{n=1}^{\infty}(n^2 + n + a_{n+2})x^n.\end{aligned}$$

Example. One more. Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Plug this into $y'' + 2y' + y$ and express it as a single power series in x.

First notice that

$$y' = \sum_{n=0}^{\infty} na_nx^{n-1} = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n.$$

And

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Therefore,

$$y'' + 2y' + y = \sum_{n=0}^{\infty} \big[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_n\big]x^n.$$

Next we apply these ideas to find recursive formulas for series solutions of some second order linear differential equations. It gets pretty tedious.

Example of Series Solution

Example 2$'$. Find a recursive formula for the terms of the series solution of $y'' + 2y' + y = 0$ with $y(0) = 1$, $y'(0) = 2$. This is the same equation as Example 2 above.

Solution. Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We plug this into $y'' + 2y' + y$ and express it as a single power series in x. Wait a minute, we just did this! The result, from the last example, is that

$$y'' + 2y' + y = \sum_{n=0}^{\infty} \big[(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_n\big]x^n = 0.$$

It follows that for each $n \geq 0$

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_n = 0.$$

We can rewrite this as

$$a_{n+2} = -\frac{2(n+1)a_{n+1} + a_n}{(n+2)(n+1)} \text{ for } n \geq 0.$$

We can shift the index and get

$$a_n = -\frac{2(n-1)a_{n-1} + a_{n-2}}{(n)(n-1)}, \text{ for } n \geq 2.$$

This is a recursive formula for a_n. Since we know $a_0 = 1$ and $a_1 = 2$ we can compute as many terms as we like.

$$\begin{aligned}
a_2 &= -\frac{2(1)a_1 + a_0}{(2)(1)} = -\frac{5}{2} \\
a_3 &= -\frac{2(2)a_2 + a_1}{(3)(2)} = -\frac{-10+2}{6} = \frac{4}{3} \\
a_4 &= -\frac{2(3)a_3 + a_2}{(4)(3)} = -\frac{8 - \frac{5}{2}}{12} = -\frac{11}{24} \\
a_5 &= -\frac{2(4)a_4 + a_3}{(5)(4)} = -\frac{-\frac{11}{3} + \frac{4}{3}}{20} = \frac{7}{60} \\
a_6 &= -\frac{2(5)a_5 + a_4}{(6)(5)} = -\frac{17}{720}
\end{aligned}$$

We can even write a short program to compute as many terms as we like.

```
> N:=20:                  # Set the number of terms to compute.
> A:= array(0..N-1):      # Define an array of length N.
> A[0]:=1:  A[1]:=2:      # Define the first two terms of the array A.
>                         # Next set up recursive formula and find the other terms.
> for n from 2 to N-1 do A[n]:= -(2*(n-1)*A[n-1] + A[n-2])/(n*(n-1)); end do;
```

The output is on the next page.

$$A_2 := -5/2$$
$$A_3 := 4/3$$
$$A_4 := -\frac{11}{24}$$
$$A_5 := \frac{7}{60}$$
$$A_6 := -\frac{17}{720}$$
$$A_7 := \frac{1}{252}$$
$$A_8 := -\frac{23}{40320}$$
$$A_9 := \frac{13}{181440}$$
$$A_{10} := -\frac{29}{3628800}$$
$$A_{11} := \frac{1}{1247400}$$
$$A_{12} := -\frac{1}{13685760}$$
$$A_{13} := \frac{19}{3113510400}$$
$$A_{14} := -\frac{41}{87178291200}$$
$$A_{15} := \frac{1}{29719872000}$$
$$A_{16} := -\frac{47}{20922789888000}$$
$$A_{17} := \frac{1}{7113748561920}$$
$$A_{18} := -\frac{53}{6402373705728000}$$
$$A_{19} := \frac{1}{2172233935872000}$$

Example

Example 5. Find a recursive formula for the series general solution to $y''+xy'+y = 0$ centered about zero.

Solution. Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then we have

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \qquad \text{and} \quad xy' = \sum_{n=0}^{\infty} n a_n x^n.$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Therefore,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

This is the same as

$$\sum_{n=0}^{\infty}\big[(n+2)(n+1)a_{n+2} + n a_n + a_n\big]x^n = 0.$$

This forces

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0 \text{ for each } n \geq 0.$$

Solving for a_{n+2} gives

$$a_{n+2} = -\frac{(n+1)a_n}{(n+2)(n+1)} = -\frac{a_n}{n+2} \text{ for each } n \geq 0.$$

We can rewrite this as

$$a_n = -\frac{a_{n-2}}{n} \text{ for each } n \geq 2.$$

Thus, if we were given a value for $y(0) = a_0$ we could find all the even a_n terms and if we were given a value for $y'(0) = a_1$ we could find all the odd a_n terms. We do this on the next page, and we find formulas for a_n as a function of n, given a_0 and a_1.

Example 5 Continued

Recall $a_n = -\dfrac{a_{n-2}}{n}$ for each $n \geq 2$. Suppose a_0 and a_1 are known. Then

$$a_2 = -a_0/2, \qquad a_4 = a_0/(2 \cdot 4), \qquad a_6 = -a_0/(2 \cdot 4 \cdot 6), \qquad a_8 = a_0/(2 \cdot 4 \cdot 6 \cdot 8), ...$$

and

$$a_3 = -a_1/3, \qquad a_5 = a_1/(3 \cdot 5), \qquad a_7 = -a_1/(3 \cdot 5 \cdot 7), \qquad a_9 = a_1/(3 \cdot 5 \cdot 7 \cdot 9), ...$$

Notice that the product of positive even numbers less than or equal to $2n$ is $2^n n!$. Thus,

$$a_{2n} = (-1)^n \frac{a_0}{2^n n!} \text{ for } n \geq 1.$$

There is no clever notation for the product of positive odd numbers less than or equal to $2n+1$, so we just write

$$a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)(2n-1)(2n-3)\cdots 3} \text{ for } n \geq 1.$$

Sometimes it is useful to consider special pairs of initial values such as $a_0 = 1$ & $a_1 = 0$ and $a_0 = 0$ & $a_1 = 1$. This is because the resulting pair of solutions will be linearly independent since their Wronskian is 1 at $x = 0$. We let

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}$$

and

$$y_2 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n-1)(2n-3)\cdots 3}.$$

These form a fundamental solution pair.

Next Example

Example 6. Find a recursive formula for the series general solution to $y''+y'+xy=0$ centered about zero.

Solution. Let $y=\sum_{n=0}^{\infty} a_n x^n$. Then we have

$$xy=\sum_{n=0}^{\infty} a_n x^{n+1}=\sum_{n=1}^{\infty} a_{n-1}x^n,$$

$$y'=\sum_{n=0}^{\infty} n a_n x^{n-1}=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n,$$

and

$$y''=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Therefore,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_n x^n+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=1}^{\infty} a_{n-1}x^n=0.$$

The summands all have the exponents of x agreeing, but they do not all start at $n=0$; one of the summations starts at $n=1$. Therefore, we have to treat the $n=0$ terms of the other summations separately. We get

$$(2a_2+a_1)+\sum_{n=1}^{\infty}\big[(n+1)na_n+(n+1)a_{n+1}+a_{n-1}\big]x^n=0.$$

Thus,

$$2a_2+a_1=0 \quad\Longrightarrow\quad a_2=-\frac{a_1}{2},$$

and

$$a_{n+1}=-\frac{(n+1)na_n+a_{n-1}}{n+1} \quad \text{for } n\geq 2.$$

The latter we could rewrite as

$$a_n=-\frac{n(n-1)a_{n-1}+a_{n-2}}{n} \quad \text{for } n\geq 3.$$

Thus, given a_0 and a_1 we can compute as many terms as we want.

Yet Another Example

Example 7. Find a recursive formula for the series general solution to $xy'' + y' + xy = 0$ centered about $x = 1$.

Solution. Let $y = \sum_{n=0}^{\infty} a_n(x-1)^n$. We use the handy fact that $x = (x-1)+1$. Then

$$xy = (x-1)\sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^n = \sum_{n=0}^{\infty} a_n(x-1)^{n+1} + \sum_{n=0}^{\infty} a_n(x-1)^n =$$

$$\sum_{n=1}^{\infty} a_{n-1}(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^n = a_0 + \sum_{n=1}^{\infty} \big[a_{n-1} + a_n\big](x-1)^n.$$

And

$$y' = \sum_{n=0}^{\infty} na_n(x-1)^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-1)^n = a_1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}(x-1)^n.$$

And

$$\begin{aligned}
xy'' &= (x-1)\sum_{n=0}^{\infty}(n-1)na_n(x-1)^{n-2} + \sum_{n=0}^{\infty}(n-1)na_n(x-1)^{n-2} \\
&= \sum_{n=2}^{\infty}(n-1)na_n(x-1)^{n-1} + \sum_{n=2}^{\infty}(n-1)na_n(x-1)^{n-2} \\
&= \sum_{n=1}^{\infty} n(n+1)a_{n+1}(x-1)^n + \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-1)^n \\
&= 2a_2 + \sum_{n=1}^{\infty}\big[n(n+1)a_{n+1} + (n+1)(n+2)a_{n+2}\big](x-1)^n.
\end{aligned}$$

Therefore,

$$xy''+y'+xy = 2a_2+a_1+a_0+\sum_{n=1}^{\infty}\big[n(n+1)a_{n+1}+(n+1)(n+2)a_{n+2}+(n+1)a_{n+1}+a_{n-1}+a_n\big](x-1)^n = 0.$$

Hence,

$$a_2 = -\frac{a_1+a_0}{2},$$

and

$$a_{n+2} = -\frac{n(n+1)a_{n+1} + (n+1)a_{n+1} + a_{n-1} + a_n}{(n+1)(n+2)} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+1)(n+2)} \quad \text{for } n \geq 1.$$

The latter can be rewritten as

$$a_n = -\frac{(n-1)^2 a_{n-1} + a_{n-2} + a_{n-3}}{(n-1)n} \quad \text{for } n \geq 3.$$

You Guessed It, Another Example!

Example 8. Find a recursive formula for the series general solution to $(4-x^2)y''+2y=0$ centered at zero.

Solution. Let $y=\sum_{n=0}^{\infty} a_n x^n$. Then

$$y''=\sum_{n=0}^{\infty}(n-1)na_nx^{n-2}=\sum_{n=2}^{\infty}(n-1)na_nx^{n-2}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n.$$

Thus,

$$x^2y''=\sum_{n=0}^{\infty}(n-1)na_nx^n.$$

Therefore,

$$(4-x^2)y''+2y=\sum_{n=0}^{\infty}\left[4(n+1)(n+2)a_{n+2}-(n-1)na_n+2a_n\right]x^n=0.$$

Thus,

$$a_{n+2}=\frac{(n-1)na_n-2a_n}{4(n+1)(n+2)}=\frac{(n-2)a_n}{4(n+2)},\quad \text{for } n\geq 0,$$

which we can rewrite as

$$a_n=\frac{(n-4)a_{n-2}}{4n},\quad \text{for } n\geq 2.$$

Mucking Around with the Previous Example

Just for fun let's consider the two cases: $a_0 = 1$ with $a_1 = 0$, and $a_0 = 0$ with $a_1 = 1$. This will give us a pair of linearly independent solutions!

In the first case, since $a_1 = 0$ we have the $a_n = 0$ for all odd n. Then

$$a_2 = \frac{2-4}{4 \cdot 2} a_0 = -\frac{1}{4} \qquad \text{and} \qquad a_4 = \frac{4-4}{4 \cdot 4} a_2 = 0.$$

It follows that $a_n = 0$ for all n even and bigger than 4. Thus,

$$y_1 = 1 - \frac{1}{4} x^2$$

gives the exact solution.

Now suppose, $a_0 = 0$ & $a_1 = 1$. Thus $a_n = 0$ for all even n. For odd n we have the following.

$$a_3 = \frac{3-4}{4 \cdot 3} a_1 = \frac{-1}{4 \cdot 3}.$$

$$a_5 = \frac{5-4}{4 \cdot 5} a_3 = \frac{-1}{4^2 \cdot 5 \cdot 3}.$$

$$a_7 = \frac{7-4}{4 \cdot 7} a_5 = \frac{-1 \cdot 3}{4^3 \cdot 7 \cdot 5 \cdot 3} = \frac{-1}{4^3 \cdot 7 \cdot 5}.$$

$$a_9 = \frac{9-4}{4 \cdot 9} a_7 = \frac{-1 \cdot 5}{4^4 \cdot 9 \cdot 7 \cdot 5} = \frac{-1}{4^4 \cdot 9 \cdot 7}.$$

Let $n = 2k + 1$. We conclude that

$$a_{2k+1} = \frac{-1}{4^k (2k+1)(2k-1)}.$$

Hence we let

$$y_2 = \sum_{n=0}^{\infty} \frac{-x^{2k+1}}{4^k (2k+1)(2k-1)}.$$

And so now we have two linearly independent solutions.

But, you know, it seems to me that we could do better. Since we have a finite term expression for y_1 we could use the **reduction of order method** to find a closed form expression for a second solution that is linearly independent from y_1. We do this on the next page.

More Mucking Around

Earlier, we found that $y_1 = 1 - \frac{1}{4}x^2$ was a solution to $(4-x^2)y'' + 2y = 0$. But so is any multiple of it, so we will work will $y_3 = -4y_1 = x^2 - 4$ to avoid fractions. Now, we use the reduction of order method. Let $y = v(x)(x^2-4)$. Then

$$y'' = (v'(x^2-4) + 2vx)' = v''(x^2-4) + 2v'x + 2v.$$

We plug this into the original differential equation to get

$$(4-x^2)\big[(x^2-4)v'' + 4xv' + 2v\big] + 2(x^2-4)v = 0.$$

Thus,

$$-(x^2-4)^2v'' - 4x(x^2-4)v' - 2(x^2-4)v + 2(x^2-4)v = 0,$$

or

$$(x^2-4)^2v'' + 4x(x^2-4)v' = 0.$$

Let $w = v'$. (Now we have **reduced the order**). We have

$$(x^2-4)^2w' + 4x(x^2-4)w = 0.$$

Thus,

$$\big((x^2-4)w\big)' = 0, \quad \Longrightarrow \quad w = \frac{C}{x^2-4}.$$

Let $C = 1$. Then we integrate to get v.

$$v = \int w\,dx = \int \frac{1}{x^2-4}\,dx = \frac{1}{4}\int \frac{1}{x-2} - \frac{1}{x+2}\,dx =$$

$$\frac{1}{4}\left(\ln|x-2| - \ln|x+2| + C\right) = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

We let $C = 0$. Also, any multiple works so we can drop the $\frac{1}{4}$. Thus, we let

$$v = \ln\left|\frac{x-2}{x+2}\right|.$$

Then we let

$$y_4 = vy_3 = (x^2-4)\ln\left|\frac{x-2}{x+2}\right|$$

Thus, $\{y_3, y_4\}$ forms a fundamental solution set.

How does y_4 relate to the series express we had for y_2? You can check that $y_4(0) = 0$ but that $y_4'(0) = 1$. Thus $y_2 = y_4$ or

$$\sum_{k=1}^{\infty} \frac{-x^{2k+1}}{4^k(2k+1)(2k-1)} = (x^2-4)\ln\left|\frac{x+2}{x-2}\right|.$$

It is a coincindence that they are equal. It coud have happened that y_4 was a constant times y_2. But, it had to be independent of y_3 which is a multiple of y_1.

Lastly, we comment on the radius of convergence.

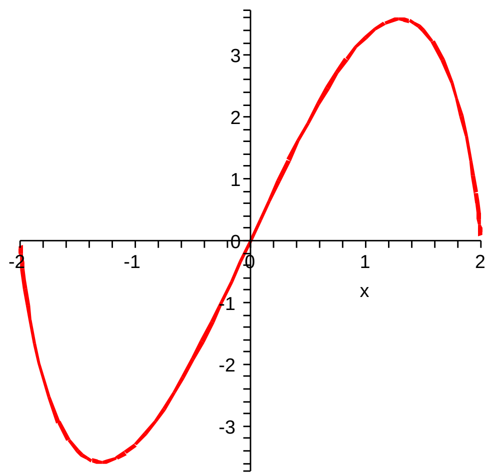
The Radius of Convergence of the Previous Example

For y_1 or y_3 the radius of convergence is clearly infinity. Putting the original problem in the form

$$y'' + \frac{2}{4-x^2}y = 0,$$

we can see that the radius of convergence in general must be at least 2. From the form of y_4 we can see that it is undefined at $x = \pm 2$. A graph of $y_4(x)$ is shown below.

If you study the limits as $x \to 2$ from below and $x \to -2$ from above you will get zero in both cases. But, the corresponding limits for $y_4'(x)$ are both $-\infty$.



Another approach is to use the ratio test directly on the series for y_2. Let's do it!

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\dfrac{x^{2k+3}}{4^{k+1}(2k+3)(2k+1)}}{\dfrac{x^{2k+1}}{4^k(2k+1)(2k-1)}}\right| = \frac{x^2}{4}\frac{2k-1}{2k+3}.$$

Now,

$$\lim_{k\to\infty} \frac{x^2}{4}\frac{2k-1}{2k+3} = \frac{x^2}{4}.$$

Thus, we have convergence for $-2 < x < 2$.

A Final Example

Example 9. Find the general series solution to $y'' + t^2 y = t^4$ centered about 0

Solution. Let $y = \sum_{n=0}^{\infty} a_n t^n$. Then

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n,$$

and

$$t^2 y = \sum_{n=0}^{\infty} a_n t^{n+2} = \sum_{n=2}^{\infty} a_{n-2} t^n.$$

Thus we have

$$y'' + t^2 y = 2a_2 + 3\cdot 2a_3 t + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} a_{n-2}t^n = t^4.$$

We subtract the t^4 from both sides. This causes us to write out the terms up to $n = 4$ and use summation notation only after that. We get

$$2a_2+3\cdot 2a_3 t+(4\cdot 3a_4+a_0)t^2+(5\cdot 4a_5+a_1)t^3+(6\cdot 5a_6+a_1-1)t^4+\sum_{n=5}^{\infty}\left[(n+2)(n+1)a_{n+2}+a_{n-2}\right]t^n = 0.$$

Now we are ready to rumble! We take a_0 and a_1 as given. We then deduce the following.

$$\begin{aligned}
2a_2 = 0 &\quad\Longrightarrow\quad a_2 = 0 \\
3\cdot 2a_3 = 0 &\quad\Longrightarrow\quad a_3 = 0 \\
4\cdot 3a_4 + a_0 = 0 &\quad\Longrightarrow\quad a_4 = \frac{-a_0}{4\cdot 3} \\
5\cdot 4a_5 + a_1 = 0 &\quad\Longrightarrow\quad a_5 = \frac{-a_1}{5\cdot 4} \\
6\cdot 5a_6 + a_2 - 1 = 0 &\quad\Longrightarrow\quad a_6 = \frac{1}{6\cdot 5} \\
(n+2)(n+1)a_{n+2} + a_{n-2} = 0 &\quad\Longrightarrow\quad a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)},\ \text{for } n \geq 5
\end{aligned}$$

The last expression can be rewritten as

$$a_n = \frac{-a_{n-4}}{n(n-1)} \text{ for } n \geq 7.$$

We are going to generate some more terms using this recursive relation and then see if we can express a_n as a function of n, for $n \geq 7$.

$$
\begin{aligned}
a_7 &= \frac{-a_3}{7\cdot 6} &&= 0\\
a_8 &= \frac{-a_4}{8\cdot 7} &&= \frac{a_0}{8\cdot 7\cdot 4\cdot 3}\\
a_9 &= \frac{-a_5}{9\cdot 8} &&= \frac{a_1}{9\cdot 8\cdot 5\cdot 4}\\
a_{10} &= \frac{-a_6}{10\cdot 9} &&= \frac{-1}{10\cdot 9\cdot 6\cdot 5}\\
a_{11} &= \frac{-a_7}{11\cdot 10} &&= 0\\
a_{12} &= \frac{-a_8}{12\cdot 11} &&= \frac{-a_0}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}\\
a_{13} &= \frac{-a_9}{13\cdot 12} &&= \frac{-a_1}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}\\
a_{14} &= \frac{-a_{10}}{14\cdot 13} &&= \frac{1}{14\cdot 13\cdot 10\cdot 9\cdot 6\cdot 5}\\
a_{15} &= \frac{-a_{11}}{15\cdot 14} &&= 0\\
a_{16} &= \frac{-a_{12}}{16\cdot 15} &&= \frac{a_0}{16\cdot 15\cdot 12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}\\
a_{17} &= \frac{-a_{13}}{17\cdot 16} &&= \frac{a_1}{17\cdot 16\cdot 13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4}
\end{aligned}
$$

Now we can finally see what is happening. For $n \geq 7$ we have

$$
a_n = \begin{cases}
\dfrac{a_0}{n(n-1)(n-4)(n-5)\cdots 4\cdot 3} & \text{for} \quad n = 0 \mod 8\\[2ex]
\dfrac{a_1}{n(n-1)(n-4)(n-5)\cdots 5\cdot 4} & \text{for} \quad n = 1 \mod 8\\[2ex]
\dfrac{-1}{n(n-1)(n-4)(n-5)\cdots 6\cdot 5} & \text{for} \quad n = 2 \mod 8\\[2ex]
0 & \text{for} \quad n = 3 \mod 8\\[2ex]
\dfrac{-a_0}{n(n-1)(n-4)(n-5)\cdots 4\cdot 3} & \text{for} \quad n = 4 \mod 8\\[2ex]
\dfrac{-a_1}{n(n-1)(n-4)(n-5)\cdots 5\cdot 4} & \text{for} \quad n = 5 \mod 8\\[2ex]
\dfrac{1}{n(n-1)(n-4)(n-5)\cdots 6\cdot 5} & \text{for} \quad n = 6 \mod 8\\[2ex]
0 & \text{for} \quad n = 7 \mod 8
\end{cases}
$$

Note: $n = k \bmod 8$ means the remainder when n is divided by 8 equals k.

When is the last time you had this much fun!