Euler's Equations

A differential equation of the form

$$at^2y''(t) + bty'(t) + cy(t) = 0, \qquad (\#)$$

is known as an **Euler Equation** or a **Cauchy-Euler Equation**. They are introduced in Problems 38-42 in Section 3.4 (8th edition) and in more detail in Sectin 5.5. See also, "Why Cauchy and Euler Share the CauchyEuler Equation," by Adam E. Parker in The College Mathematics Journal, Vol. 47, No. 3, May 2016.

We assume we are interested in solutions for which $t < 0$ or $t > 0$. Thinking ourselves clever we suppose $y = t^r$ is a solution. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substitution gives

$$ar(r-1)t^r + brt^r + ct^r = 0.$$

We divide through by t^r and simplify to get

$$ar^2 + (b-a)r + c = 0.$$

Now we have three cases, the roots are real and distinct, there is one real root, or the roots are complex conjugates.

Case 1. Suppose the roots, r_1 and r_2, are real and distinct. Then t^{r_1} and t^{r_2} are solutions, and so too would be any linear combination. You can check that they are linearly independent. Thus the general solution is

$$y(t) = C_1t^{r_1} + C_2t^{r_2}.$$

Example 1. Solve $t^2y'' + 5ty'' - 5y = 0$, with initial conditions $y(1) = 2$, $y'(1) = 7$.

Solution. The characteristic equation is

$$r^2 + (5-1)r - 5 = r^2 + 4r - 5 = (r+5)(r-1).$$

The roots are -5 and 1. Thus, the general solution is

$$y(t) = C_1t^{-5} + C_2t.$$

The initial conditions give $C_1 + C_2 = 1$ and $-5C_1 + C_2 = 7$. Thus, $C_1 = -1$ and $C_2 = 2$. The solution for the given initial conditions is therefore,

$$y(t) = -t^{-5} + 2t.$$

□

Case 2. Suppose r is the only root. Then t^r is a solution. To get a second linearly independent solution we use the reduction of order method. Let $y = vt^r$. Then

$$y' = v't^r + rvt^{r-1} \ \& \ y'' = v''t^r + 2rvt^{r-1} + r(r-1)vt^{r-2}.$$

Substitution into (#), after simplifying, gives

$$at^{r+2}v'' + (2ar+b)v' = 0.$$

We let $w = v'$, and place the result into standard form to get

$$w' + \frac{2ar+b}{at}w = 0. \qquad (\%)$$

Now, since r is the only root we know from the quadratic formula that $r = (a-b)/2a$. Thus,

$$\frac{2ar+b}{a} = \frac{2a\frac{a-b}{2a}+b}{a} = \frac{a-b+b}{a} = 1.$$

Thus, (%) becomes

$$w' + \frac{1}{t}w = 0 \implies tw' + w = 0 \implies (tw)' = 0 \implies w = C/t.$$

We let $C = 1$. Then

$$v = \int w\,dt = \int \frac{1}{t}\,dt = \ln|t| + C$$

Let $C = 0$. Then $y = t^r \ln|t|$. You can check that t^r and $t^r \ln|t|$ are linearly independent. Thus the general solution for this case is

$$y(t) = C_1 t^r + C_2 t^r \ln|t|.$$

We can drop the absolute value symbols if we know $t > 0$.

Example 2. Find the general solution to $t^2y'' - ty' + y = 0$.

Solution. The characteristic polynomial is

$$r^2 + (-1-1)r + 1 = r^2 - 2r + 1 = (r-1)^2.$$

It has 1 as a repeated root. Therefore, the general solution is

$$y(t) = C_1 t + C_2 t \ln|t|.$$

□

Case 3. Let $z = p + iq$ be a complex number. Assume $t > 0$. Then

$$t^z = e^{\ln(t^z)} = e^{z\ln t} = e^{p\ln t}\big(\cos(q\ln t) + i\sin(q\ln t)\big) =$$

$$t^p\big(\cos(q\ln t) + i\sin(q\ln t)\big).$$

Suppose the roots of the characteristic polynomial are $\alpha \pm i\beta$. Then the general solution is

$$\begin{aligned} y(t) &= C_1 t^{\alpha+i\beta} + C_1 t^{\alpha-i\beta} \\ &= C_1 t^\alpha\left(\cos(\beta\ln t) + i\sin(\beta\ln t)\right) + C_2 t^\alpha\left(\cos(\beta\ln t) - i\sin(\beta\ln t)\right) \\ &= (C_1 + C_2)t^\alpha\cos(\beta\ln t) + (C_1 - C_2)it^\alpha\sin(\beta\ln t). \end{aligned}$$

This can be rewritten as

$$y(t) = At^\alpha\cos(\beta\ln t) + Bt^\alpha\sin(\beta\ln t).$$

If the initial conditions are real, then A and B will be real. If the initial conditions are at a negative value of t, replace each $\ln t$ with $\ln|t|$.

Example 3. Find the general solution to $t^2y'' - ty' + 2y = 0$, and then find the particular solution for $y(1) = 0$, $y'(1) = 1$.

Solution. The characteristic polynomial is

$$r^2 + (-1-1)r + 2 = r^2 - 2r + 2.$$

Check that the roots are the complex numbers $1 \pm i$. Therefore, the general solution is

$$y(t) = C_1 t\cos(\ln t) + C_2 t\sin(\ln t).$$

Now,

$$y(1) = C_1 \cos(\ln 1) + C_2 \sin(\ln 1) = C_1 \cos(0) + C_2 \sin(0) = C_1.$$

Thus, $C_1 = 0$. Next

$$y'(1) = C_2 \sin \ln 1 + C_2 \frac{\cos \ln 1}{1} = C_2.$$

Thus, $C_2 = 1$ and our solution is

$$y(t) = t \sin(\ln t).$$

□

Summary

Case 1. The roots of the characteristic polynomial, r_1 and r_2, are real and distinct, that is $r_1 \neq r_2$. Then the general solution is

$$y(x) = C_1 t^{r_1} + C_2 t^{r_2}.$$

Case 2. The characteristic polynomial has a single real root, r. Then the general solution is

$$y(x) = C_1 t^r + C_2 t^r \ln |t|.$$

Case 3. The roots of the characteristic polynomial are complex conjugates, $\alpha \pm i\beta$. Then the general solution is

$$y(x) = C_1 t^\alpha \cos(\beta \ln |t|) + C_2 t^\alpha \sin(\beta \ln |t|).$$

In each case we can find unique values of C_1 and C_2 for any given pair of initial conditions of the form

$$y(t_0) = y_0 \quad \& \quad y'(t_0) = v_0,$$

provided $t_0 \neq 0$.

Student Exercises. Solve the following. Assume $t > 0$.

(1) $t^2 y'' + 7ty' + 9y = 0$, $y(e) = 1$, $y'(e) = 0$.
(2) $2t^2 y'' - 5ty' + 3y = 0$, $y(1) = 1$, $y'(1) = 2$.
(3) $t^2 y'' + 2ty' - 2y = 0$, $y(1) = 1$, $y'(1) = 1$.
(4) $t^2 y'' - 3ty' + 13y = 0$, $y(1) = 1$, $y'(1) = 0$.

Answers.
(1) $y(t) = e^3(3\ln(t) - 2)/t^3$.
(2) $y(t) = (3t^3 + 2t^{1/2})/5$.
(3) $y(t) = t$.
(4) $y(t) = t^2(3\cos(3\ln(t)) - 2\sin(3\ln(t)))/3$.