Lecture Notes for Ch 10
Fourier Series and Partial Differential Equations[1]

Contents

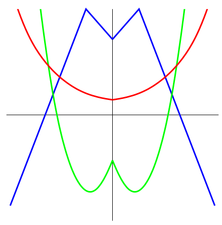
1 Review of Even and Odd Functions

Even: $f(-x) = f(x)$	Odd: $f(-x) = -f(x)$.

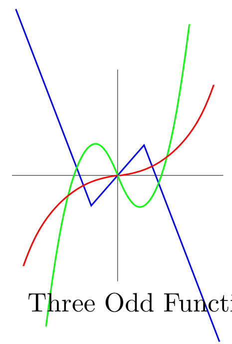
Even Examples: $5x^6 - 3x^4 + 2$, $\frac{x^4+1}{x^2-7}$, $|x|$, $\cos(x)$, $\cos(x^5)$, $\sin^4(x)$.

Odd Examples: $x^{1/3}$, $x^7 - 6x^3$, $x|x|$, $\sin(x)$, $\sin^5(x^7)$.

Neither: $x^2 + x$, $x + \cos(x)$, $\frac{1}{x+1}$.



Three Even Functions



Three Odd Functions

Problem: Let $f(x)$ be an odd function and suppose it is defined at $x = 0$. What is $f(0)$? Prove this!

Problem: Find a function with domain the whole real line that is both even and odd. (There is only one correct answer.)

Fact: If $f(x)$ is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Fact: If $f(x)$ is odd, then $\int_{-a}^{a} f(x)\,dx = 0$. Draw pictures to see intuitively why these two facts hold.

Fact: If $f(x)$ is even and differentiable, then $f'(x)$ is odd, and vice versa. This can be proved with the Chain Rule. Suppose $f(x)$ is even and differentiable. Then $(f(-x))' = f'(-x)(-1)$. But $f(-x) = f(x)$, so $(f(-x))' = (f(x))' = f'(x)$. Thus, $f'(-x)(-1) = f'(x)$, or $f'(-x) = -f'(x)$. You can also draw pictures of tangent lines to curves to see this intuitively.

Exercises: Suppose that $f(x)$ and $g(x)$ are even and the $h(x)$ and $k(x)$ are odd. Then show that:

(a) $f(x)g(x)$ is even.	(f) $f(g(x))$ is even.
(b) $f(x)h(x)$ is odd.	(g) $f(h(x))$ is even.
(c) $h(x)k(x)$ is even.	(h) $h(f(x))$ is even.
(d) $f(x) + g(x)$ is even.	(i) $h(k(x))$ is odd.
(e) $h(x) + k(x)$ is odd.	(j) $f(x) + h(x)$ need not be even or odd.

Example: Let $p(x) = f(x)h(x)$. Then $p(-x) = f(-x)h(-x) = f(x)(-h(x)) = -f(x)h(x) = -p(x)$. Hence we have an odd function.

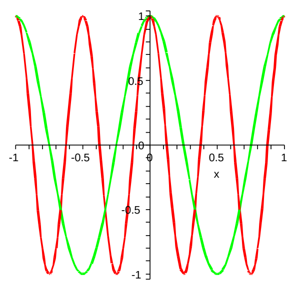
2 Some Damn Useful Integral Formulas

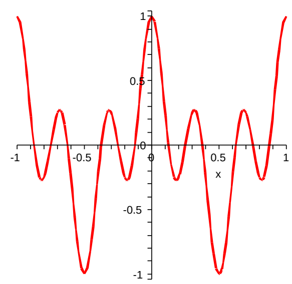
1. $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0 & \text{for} \quad m \neq n, \\ L & \text{for} \quad m = n. \end{cases}$

2. $\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\, dx = 0$, for all integers m and n

3. $\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\, dx = \begin{cases} 0 & \text{for} \quad m \neq n, \\ L & \text{for} \quad m = n. \end{cases}$

Below and to the left are the overlaid plots of $\cos 4\pi x$ and $\cos 2\pi x$. Below and to the right is the graph of their product. Study this. Make some similar plots on your own until the formulas above make sense.





We will prove only a special case of #1. Let $L = \pi$, $m = 3$ and $n = 2$. The proof uses the trig identity

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$

and its corollary

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Thus,

$$\cos\theta\cos\phi = \frac{1}{2}\left(\cos(\theta + \phi) + \cos(\theta - \phi)\right).$$

Applying this to our case gives

$$\cos 3x \cos 2x = \frac{1}{2}\left(\cos 5x + \cos x\right).$$

Thus,

$$\int_{-\pi}^{\pi} \cos 3x \cos 2x\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos 5x + \cos x\, dx = 0 + 0 = 0.$$

Let's consider one more special case: $L = \pi$, $m = n = 2$. Then

$$\int_{-\pi}^{\pi} \cos^2 2x\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos 4x + \cos 0\, dx = \frac{0 + 2\pi}{2} = \pi.$$

From these ideas you should be able to derive the three integrals formulas. If you have had linear algebra, you might notice that the integral of a product of two functions is a kind of inner product and so the cosine and sine functions used above are mutually orthogonal.

3 Boundary Value Problems

In Chapter 3 we studied Initial Value Problems that took the form

$$ay''(t) + by'(t) + cy(t) = 0 \qquad y(t_0) = p \ \& \ y'(t_0) = q.$$

These always had a unique solution.

We will now study Boundary Value Problems. Suppose we are only interested in finding a solution to

$$ay''(t) + by'(t) + cy(t) = 0$$

on some interval, $[t_1, t_2]$ and we know the value of y at the end points,

$$y(t_1) = p \ \& \ y(t_2) = q.$$

We do some examples.

Example 1. Find all solutions to

$$y'' + y = 0 \qquad y(0) = 1 \ \& \ y(\pi) = 2.$$

or show there are none.

Solution. The general solution is $y = C_1 \sin t + C_2 \cos t$. Now $y(0) = C_2$, so $C_2 = 1$. But $y(\pi) = -C_2$, so $C_2 = -2$. Since $1 \neq -2$ there are no solutions.

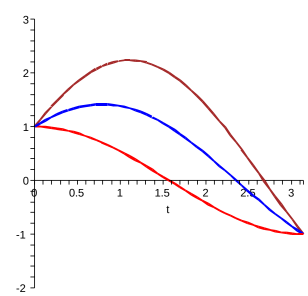
Example 2. Find all solutions to

$$y'' + y = 0 \qquad y(0) = 1 \ \& \ y(\pi) = -1.$$

or show there are none.

Solution. In this case $C_2 = 1$ work at both end points. Notice that C_1 can take on any value. Thus, there are infinity many valid solutions. Below, we plot three of these.

```
> plot([cos(t),sin(t) + cos(t),2*sin(t)+cos(t)],t=0..Pi,view=[0..Pi,-2..3],
color=[red,blue,brown],thickness=2);
```



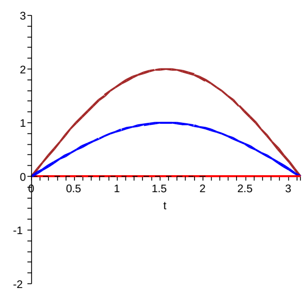
Example 3. Find all solutions to

$$y'' + y = 0 \qquad y(0) = 0 \;\&\; y(\pi) = 0.$$

or show there are none.

Solution. In this case $C_2 = 0$ works for both end points. But agian C_1 can take on any value. Thus, there are infinity many valid solutions. Below, we plot three of these.

```
> plot([0,sin(t),2*sin(t)],t=0..Pi,view=[0..Pi,-2..3],
color=[red,blue,brown],thickness=2);
```



Example 4. Consider $y'' + \gamma y = 0$, $y(0) = y(\pi) = 0$. For which values of γ do nontrivial solutions exist? By *nontrivial* we mean that $y(t)$ is not all always equal to zero, that is, we regrad $y(t) = 0$ as the *trivial solution.*

Solution. The characteristic polynomial is $r^2 + \gamma = 0$. So $r = \pm\sqrt{-\gamma}$. There are three cases. (I) What if $\gamma < 0$? The values of r are real and distinct. The general solution is

$$y(t) = C_1 e^{t\sqrt{-\gamma}} + C_2 e^{-t\sqrt{-\gamma}}$$

You can show that this forces $C_1 = C_2 = 0$.

(II) Suppose $\gamma = 0$. Then the general solution is $C_1 t + C_2$. Again the boundary values force $C_1 = C_2 = 0$.

(III) Now $\gamma > 0$. Now the roots of the characteristic polynomial are imaginary, $r = \pm i\sqrt{\gamma}$. Thus, we can write the general solution as

$$y(t) = C_1 \sin t\sqrt{\gamma} + C_2 \cos t\sqrt{\gamma}$$

Now $y(0) = C_2$, so $C_2 = 0$. The other boundary value gives $C_1 \sin \pi\sqrt{\gamma} = 0$. Unless $\sqrt{\gamma}$ is an integer, this forces $C_1 = 0$. Therefore, the allowed values of γ are 1, 4, 9, 16, 25, 36, ...

4 Definition of Fourier Series

Let $f(x)$ be a piecewise continuous periodic function with period $2L$.[2] The Fourier Series of $f(x)$ is,

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx, \quad n = 0, 1, 2, 3, \ldots,$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx, \quad n = 1, 2, 3, \ldots.$$

Theorem. The Fourier series of $f(x)$ converges to $f(x)$ if f is continuous at x. If f is discontinuous at x then this must be a jump discontinuity. Let

$$f(x^+) = \lim_{c \to x^+} f(c) \qquad \text{and} \qquad f(x^-) = \lim_{c \to x^-} f(c).$$

Then the Fourier series of f converges to the average of these two limits,

$$\frac{f(x^+) + f(x^-)}{2}.$$

If f is an even function then $b_n = 0$, for $n = 1, 2, 3, \ldots$.
If f is an odd function then $a_n = 0$, for $n = 0, 1, 2, 3, \ldots$.
In all cases $a_0/2$ is the average value of $f(x)$ over one period.

We will verify parts of this theorem. The full proof is covered in Math 407.

4.1 We verify the formula for a_0 and show it is the average value of f(x).

Let $c_n = \cos\left(\frac{n\pi x}{L}\right)$ and $s_n = \sin\left(\frac{n\pi x}{L}\right)$.

Suppose

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} b_n s_n.$$

We will show that

$$a_0 = \frac{1}{L}\int_{-L}^{L} \cos(0) f(x)\,dx.$$

[2]Technical note: It is also required that $f(x)$ is bounded and that in each period of $f(x)$ there are only finitely many extrema.

$$
\begin{aligned}
\frac{1}{L}\int_{-L}^{L}\cos(0)f(x)\,dx &= \frac{1}{L}\int_{-L}^{L}1\cdot f(x)\,dx \\
&= \frac{1}{L}\int_{-L}^{L}\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nc_n+\sum_{n=1}^{\infty}b_ns_n\right)dx \\
&= \frac{1}{L}\int_{-L}^{L}\frac{a_0}{2}\,dx+\frac{1}{L}\sum_{n=1}^{\infty}a_n\int_{-L}^{L}c_n\,dx+\frac{1}{L}\sum_{n=1}^{\infty}b_n\int_{-L}^{L}s_n\,dx \\
&= a_0+\frac{1}{L}\sum_{n=1}^{\infty}a_n\cdot 0+\frac{1}{L}\sum_{n=1}^{\infty}b_n\cdot 0=a_0
\end{aligned}
$$

since,

$$
\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx=\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\bigg|_{-L}^{L}=0-0=0,
$$

$$
\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)dx=\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\bigg|_{-L}^{L}=\frac{-L}{n\pi}\left(\cos(n\pi)-\cos(-n\pi)\right)=0,
$$

and

$$
\frac{1}{L}\int_{-L}^{L}\frac{a_0}{2}\,dx=\frac{a_0}{2L}\int_{-L}^{L}1\,dx=\frac{a_0}{2L}\cdot 2L=a_0.
$$

Finally, recall that the average value of $f(x)$ over a cycle is $\frac{1}{2L}\int_{-L}^{L}f(x)\,dx$. Thus, $a_0/2$ is the average value of $f(x)$ over a cycle.

4.2 We check the formula for a_7, but the method is the same for all $n \geq 1$.

$$
\begin{aligned}
a_7 &\stackrel{?}{=} \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{7\pi x}{L}\right)f(x)\,dx \\
&= \frac{1}{L}\int_{-L}^{L}c_7\left(\frac{a_0}{2}+\sum_{n=1}^{\infty}a_nc_n+\sum_{n=1}^{\infty}b_ns_n\right)dx \\
&= \frac{a_0}{2L}\int_{-L}^{L}c_7\,dx+\frac{1}{L}\sum_{n=1}^{\infty}a_n\int_{-L}^{L}c_nc_7\,dx+\frac{1}{L}\sum_{n=1}^{\infty}b_n\int_{-L}^{L}s_nc_7\,dx \\
&= \frac{a_0}{2L}\cdot 0+\frac{1}{L}(0+0+0+0+0+0+a_7L+0+0+0+\cdots)+\frac{1}{L}(0+0+0+\cdots) \\
&= a_7.
\end{aligned}
$$

4.3 Next we check b_4

.

$$\begin{aligned}
b_4 &\stackrel{?}{=} \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{4\pi x}{L}\right) f(x)\,dx \\
&= \frac{1}{L}\int_{-L}^{L} s_4 \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n c_n + \sum_{n=1}^{\infty} b_n s_n\right) dx \\
&= \frac{a_0}{2L}\int_{-L}^{L} s_4\,dx + \frac{1}{L}\sum_{n=1}^{\infty} a_n \int_{-L}^{L} c_n s_4\,dx + \frac{1}{L}\sum_{n=1}^{\infty} b_n \int_{-L}^{L} s_n s_4\,dx \\
&= \frac{a_0}{2L}\cdot 0 + \frac{1}{L}(0+0+0+\cdots) + \frac{1}{L}(0+0+0+b_4 L + 0+0+0+\cdots) \\
&= b_4.
\end{aligned}$$

4.4 Even and Odd Properties of Fourier Series

If $f(x)$ is an odd function then

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx = 0,$$

since the product of an even function with an odd function is odd.

If $f(x)$ is an even function then

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx = 0,$$

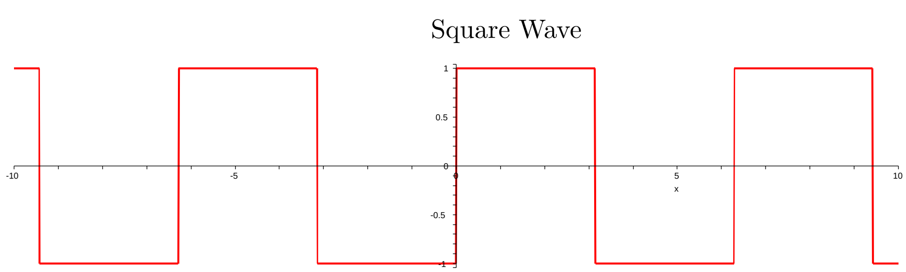
since the product of an odd function with an even function is odd.

5 An Example: The Square Wave

Let

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < 0 \\ 1 & \text{for} \quad 0 < x < \pi, \end{cases}$$

with $f(x+2\pi) = f(x)$ for all x. See the graph below. Note that the vertical lines are not part of the true graph, but are an artifact of the graphing program. I could have removed them, but they do show why this function is called a *Square Wave*.



Problem. Find the Fourier series of $f(x)$.

Solution. Since $f(x)$ is odd, $a_n = 0$ for $n = 0, 1, 2, \ldots$.

Next,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx) f(x), dx = \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\cdot 1\, dx =$$
$$\frac{-2}{n\pi}\left(\cos(n\pi) - \cos(0)\right) = \begin{cases} 0 & \text{for} \quad n \text{ even} \\ \frac{4}{n\pi} & \text{for} \quad n \text{ odd.} \end{cases}$$

Thus,

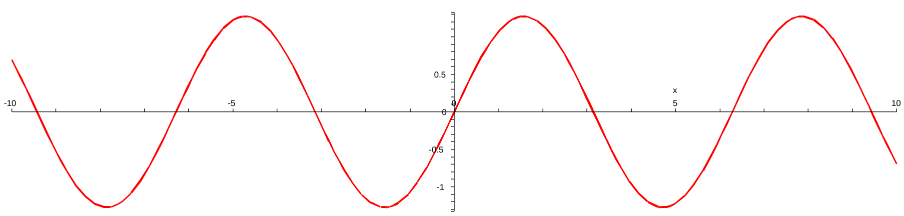
$$f(x) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \sin\left((2k-1)x\right) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \cdots .$$

Recall that using $(2k-1)$ is a way to generate only odd numbers.

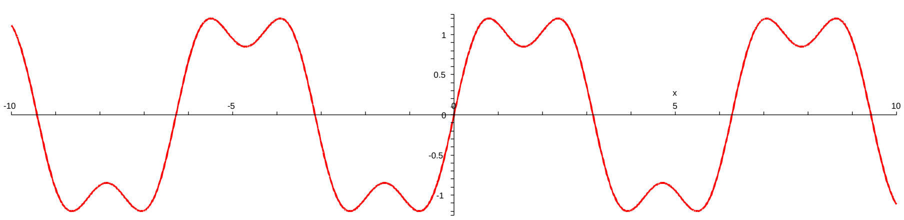
Next we plot a few partial sums. Let $f_N(x) = \sum_{k=1}^{N} \frac{4}{(2k-1)\pi}\sin\left((2k-1)x\right)$.

Plots of $f_N(x)$ for $N = 1, 2, 3, 4, 10, 100$

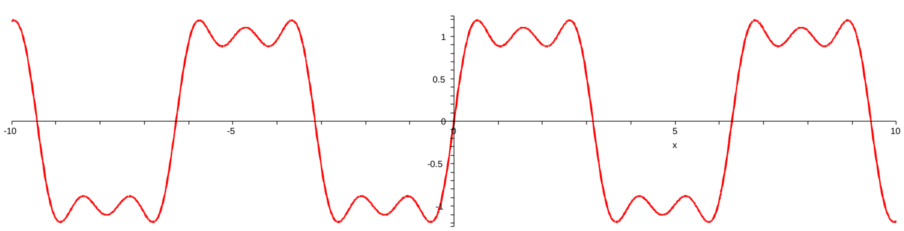
$N = 1$



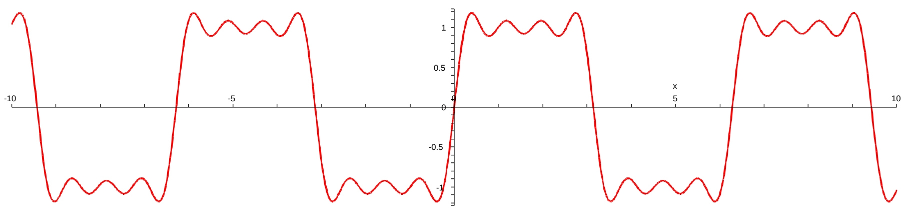
$N = 2$



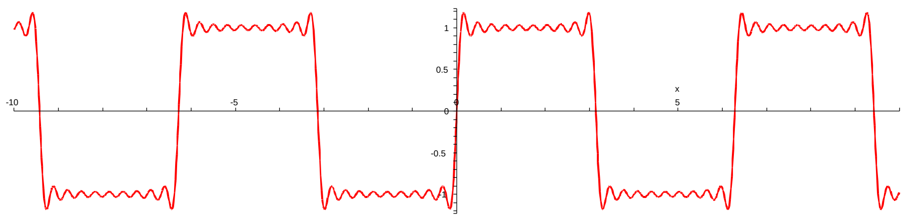
$N = 3$



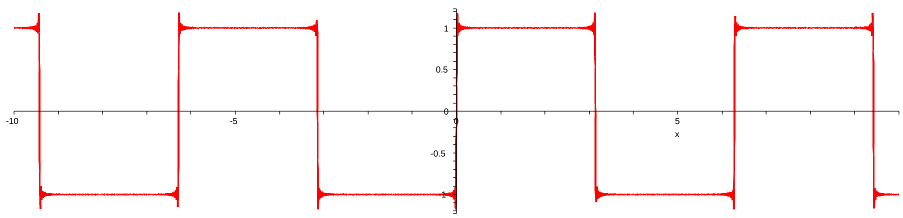
$N = 4$



$N = 10$

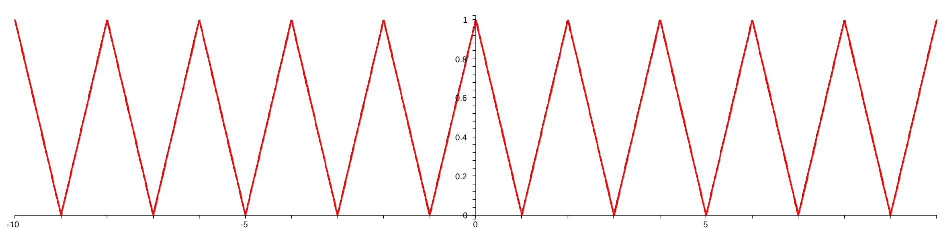


$N = 100$



6 Another Example! A Triangle Wave

Problem. Find the Fourier series of the wave depicted below.



Solution. First note that since the period is 2, $L = 1$. We can see that for $0 < x < 1$, $f(x) = 1 - x$.

Since this function is even, $b_n = 0$ for $n = 1, 2, 3, \ldots$. Also $\frac{a_0}{2}$ = ave. value of $f(x) = \frac{1}{2}$. Thus, $a_0 = 1$.

Now,

$$\begin{aligned}
a_n &= \frac{1}{1}\int_{-1}^{1} \cos(n\pi x) f(x)\, dx \\
&= 2\int_{0}^{1} \cos(n\pi x)(1-x)\, dx \\
&= \frac{2}{n\pi}\left(\sin(n\pi x) - x\sin(n\pi x) - \frac{1}{n\pi}\cos(n\pi x)\right)\Bigg|_0^1 \\
&= \frac{2}{n\pi}\left[\left(-\frac{1}{n\pi}\cos(n\pi)\right) - \left(-\frac{1}{n\pi}\right)\right] \\
&= \begin{cases} 0 & \text{for even } n \\ \frac{4}{n^2\pi^2} & \text{for odd } n. \end{cases}
\end{aligned}$$

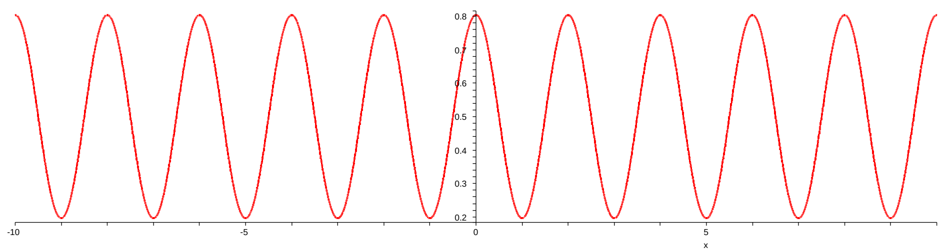
Thus,

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} \cos\left((2k-1)\pi x\right).$$

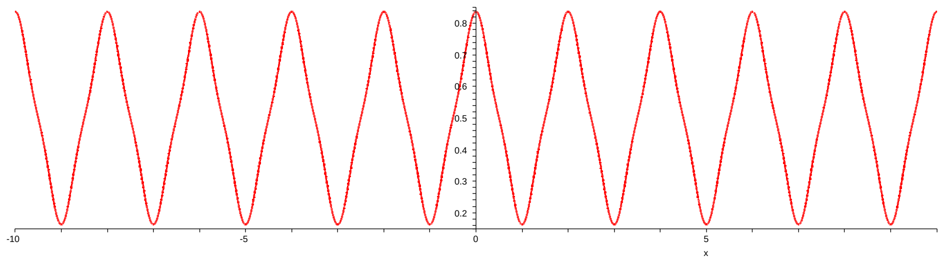
Next we plot a few partial sums. Let $f_N(x) = \frac{1}{2} + \sum_{k=1}^{N} \frac{4}{(2k-1)^2\pi^2} \cos\left((2k-1)\pi x\right)$.

Plots of $f_N(x)$ for $N = 1, 2, 3, 30$

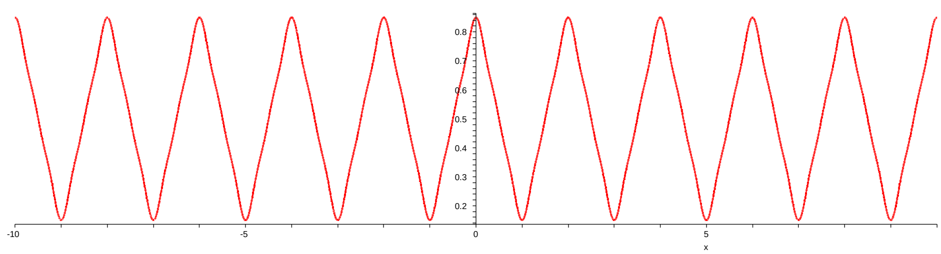
$N = 1$



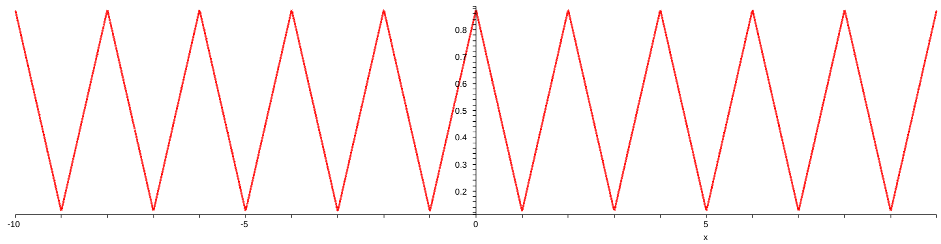
$N = 2$



$N = 3$



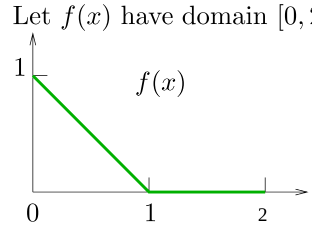
$N = 30$



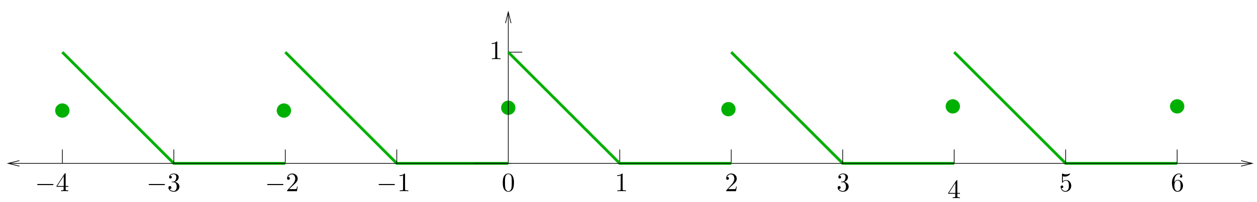
7 Periodic Extensions

We will be interested in functions on a closed bounded interval. For example the temperature along a finite metal bar. To apply Fourier series, we create a periodic extension. There are **three** standard ways to do this. We will illustrate with an example. Let $f(x)$ have domain $[0, 2]$ and be given by

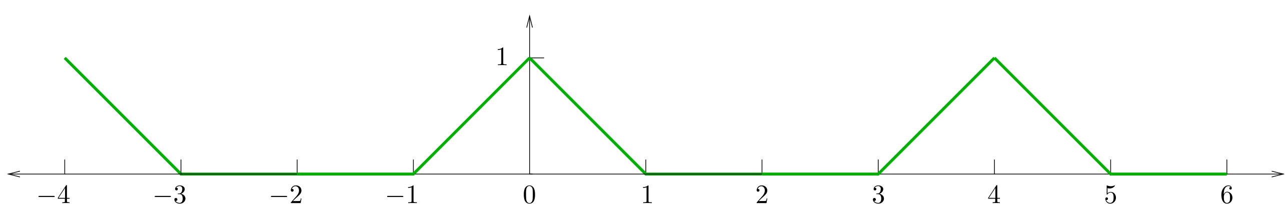
$$f(x) = \begin{cases} 1 - x & \text{for} \quad x \in [0, 1], \\ 0 & \text{for} \quad x \in (1, 2]. \end{cases}$$



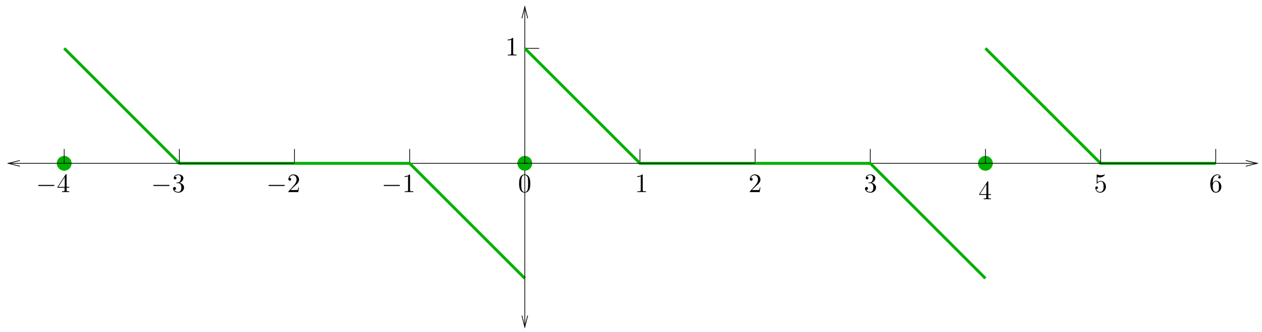
Then the **periodic extension** of $f(x)$ is the function shown below. It is given by $\tilde{f}(x) = f(x \mod 2)$ except for $x = 2n$ where jump discontinuities form. Here we define it to be the average of the two point end values.



Next we form the **even periodic extension** of $f(x)$. We first define a function on $[-2, 2]$ by letting $f_e(x) = f(|x|)$, and then forming it periodic extension, $\tilde{f}_e(x) = f_e(x \mod 4)$. In this case there are no discontinuities. If there were we would define the extension at such points as before. See below. Notice, the period is 4.



The **odd periodic extension** is similar. Let $f_o(x) = \text{sign}(x) f(|x|)$ for $x \in [0, 4]$, then extend to a function on the real line with period 4. See below.



8 Example of Fourier Series of Periodic Extensions

Problem. Find a Fourier series that will converge to $f(x) = \begin{cases} 1-x & \text{for} \quad x \in [0,1], \\ 0 & \text{for} \quad x \in (1,2] \end{cases}$ for $x \in [0,2]$.

Solution. We will find the Fourier series of its even periodic extension. Clearly, the b_n's are all zero. Let $L = 2$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f_e(x)\,dx = \frac{1}{2}\int_{-2}^{2} f_e(x)\,dx = \frac{1}{2} \times \text{area under the function} = \frac{1}{2} \times 1 = \frac{1}{2}.$$

Next,

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} \cos\left(\frac{n\pi x}{2}\right) f_e(x)\,dx \\
&= \frac{2}{2}\int_{0}^{2} \cos\left(\frac{n\pi x}{2}\right) f_e(x)\,dx \\
&= \int_{0}^{1} \cos\left(\frac{n\pi x}{2}\right)(1-x)\,dx + 0 \\
&= \int_{0}^{1} \cos\left(\frac{n\pi x}{2}\right)\,dx - \int_{0}^{1} x\cos\left(\frac{n\pi x}{2}\right)\,dx \\
&= \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{2}\right) + \frac{n\pi x}{2}\sin\left(\frac{n\pi x}{2}\right)\right]\Big|_0^1 \\
&= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\sin\left(\frac{n\pi}{2}\right) - 1\right] \\
&= \frac{-4}{n^2\pi^2}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right].
\end{aligned}$$

If you plug in some values of n you will see that

$$\cos\left(\frac{n\pi}{2}\right) - 1 = \begin{cases} -1 & \text{for} \quad n = 1 \mod 4 \\ -2 & \text{for} \quad n = 2 \mod 4 \\ -1 & \text{for} \quad n = 3 \mod 4 \\ 0 & \text{for} \quad n = 0 \mod 4 \end{cases}$$

We give the first few values of a_n, $n > 0$, in the table below.

n	1	2	3	4	5	6	7	8
a_n	$\frac{4}{\pi^2}$	$\frac{8}{4\pi^2}$	$\frac{4}{9\pi^2}$	0	$\frac{4}{25\pi^2}$	$\frac{8}{36\pi^2}$	$\frac{4}{49\pi^2}$	0

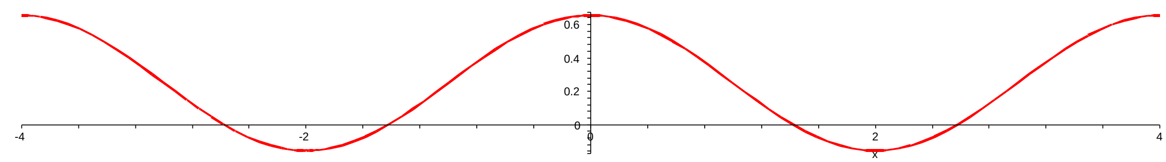
Therefore,

$$f(x) = \frac{1}{4} + \frac{4}{\pi^2}\cos\left(\frac{\pi x}{2}\right) + \frac{8}{4\pi^2}\cos\left(\frac{2\pi x}{2}\right) + \frac{4}{9\pi^2}\cos\left(\frac{3\pi x}{2}\right) + 0 + \frac{4}{25\pi^2}\cos\left(\frac{5\pi x}{2}\right) + \frac{8}{36\pi^2}\cos\left(\frac{6\pi x}{2}\right) + \cdots$$

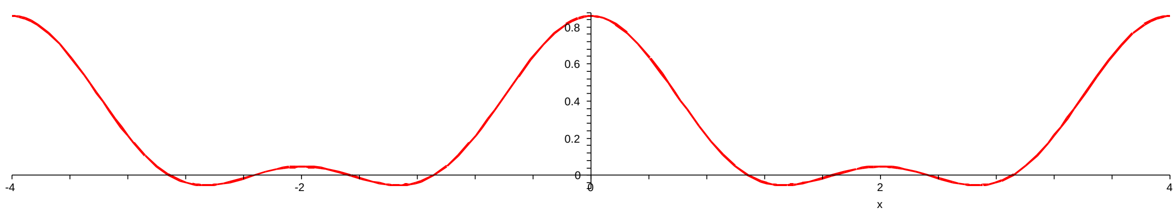
Plots of some partial sums are below.

Plots of some Partial Sums

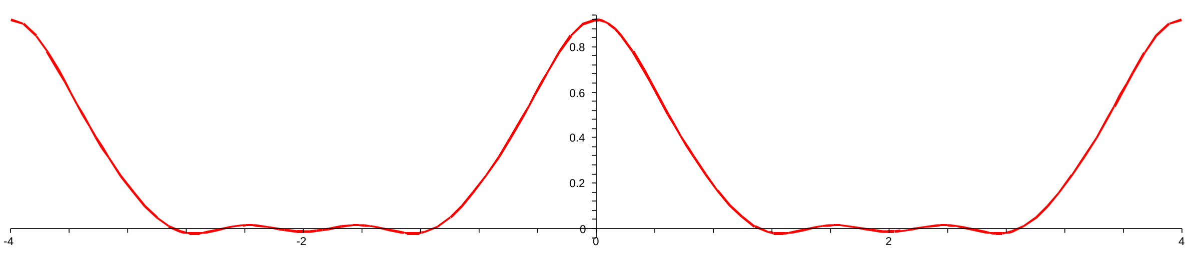
$N = 1$



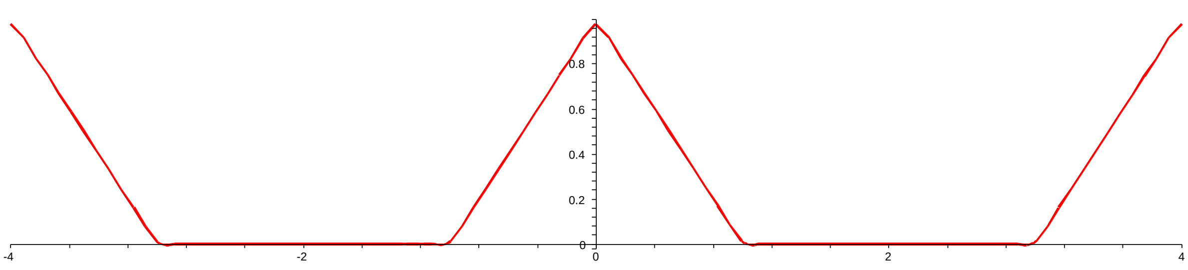
$N = 2$



$N = 5$

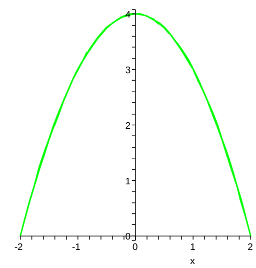


$N = 20$



9 An Example Done Entirely on Maple

Problem. Find the Fourier series that will converge to $f(x) = 4 - x^2$ for $x \in [-2, 2]$.



Solution. Since $f(x)$ is even $b_n = 0$. Next we compute a_0.

```
1/2*int( 4-x^2, x=-2..2);
```

$$\frac{16}{3}$$

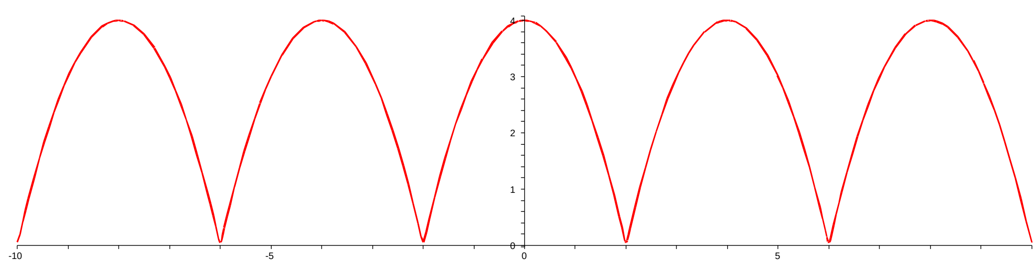
Then we find a_n.

```
> 1/2*int( (4-x^2)*cos(n*Pi*x/2), x=-2..2);
```

$$-\frac{16(n\pi\cos(n\pi) - \sin(n\pi))}{n^3\pi^3}$$

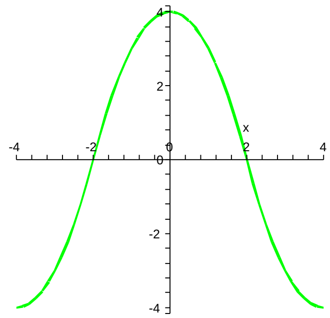
Then we plot a partial sum of the first 11 terms. Note that I simplified the expression for a_n.

```
> plot(8/3 - 16/Pi^2*sum((-1)^n/n^2*cos(n*Pi*x/2),n=1..10),x=-10..10,thickness=2);
```



9.1 Extra Credit!

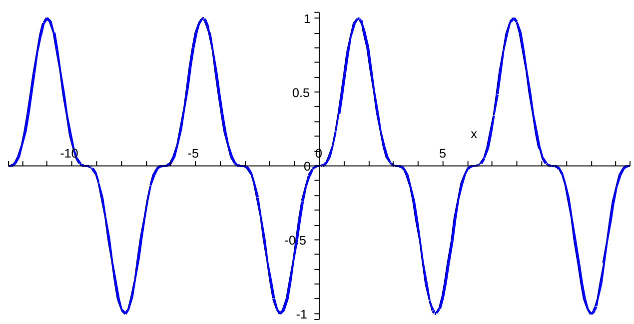
Looking at the graph from the last example, we notice it never quite gets to zero. We begin think, could there be a better way? The periodic extension of the parabola has very sharp corners. These make convergence "difficult". To get around this, use the periodic extension of a modified graph shown below.



The function is based on the original curve $y = 4 - x^2$ for $x \in [-2, 2]$, but for $x \in [-4, 2]$ and for $x \in [2, 4]$ we have taken half of original curve flipped it twice and moved it into place. Find the equations for these pieces of the new function. Take its periodic extension, find its Fourier series and compare the convergence with the Fourier series used in the last example.

10 An Example that Requires no Computer Skills

Problem. Find the Fourier series for $f(x) = \sin^3 x$.



Solution. You did this for homework already! Remember when you showed that

$$\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x.$$

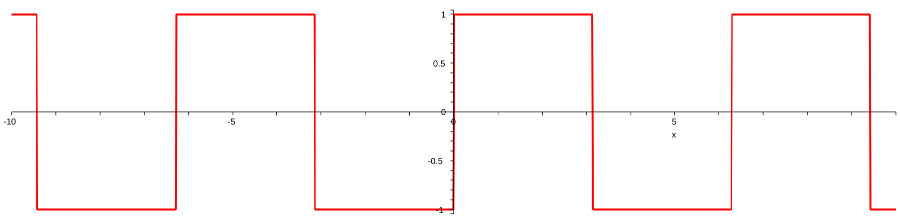
That is its Fourier series. It is odd, so all the a_n's are zero. Then $b_1 = 3/4$, $b_3 = -1/4$ and all the other b_n's are zero.

Conclusion. Trigonometry is useful after all! Who would have guessed it! Call or write your trigonometry teacher and thank her or him!

11 An ODE with Square Wave Forcing Example

Let $f(x)$ be the square wave we studied earlier, shown again below.

Square Wave



Problem. Solve the initial value problem $u'' + \frac{1}{4}u = f(x)$, $u(0) = u'(0) = 0$.

Solution. The solution to the corresponding homogeneous problem is $u_h = C_1 \sin(x/2) + C_2 \cos(x/2)$.
To get a particular solution we recall the Fourier series we found for $f(x)$:

$$\sum_{n=1}^{\infty} b_n \sin(nx), \qquad \text{where } b_n = \begin{cases} 0 & \text{for } n \text{ even,} \\ \frac{4}{n\pi} & \text{for } n \text{ odd.} \end{cases}$$

This converges to $f(x)$ except at the jump discontinuities. The idea is, we can consider each term separately. That is for $n = 1, 2, 3, \ldots$ we have

$$u'' + \frac{u}{4} = \sin nx.$$

For each n we shall find a particular solution u_n. Let

$$u_n = A \sin nx + B \cos nx.$$

Then

$$u_n'' = -An^2 \sin nx - Bn^2 \cos nx.$$

Thus, $-An^2 + A/4 = 1$ and $-Bn^2 + B/4 = 0$. Thus, $B = 0$ and $A = \dfrac{1}{\frac{1}{4} - n^2}$. Hence,

$$u_n = \frac{1}{\frac{1}{4} - n^2} \sin nx, \qquad \text{for } n = 1, 2, 3, \ldots.$$

Now we let

$$u_p = \sum_{n=1}^{\infty} \frac{b_n}{\frac{1}{4} - n^2} \sin nx.$$

It follows by linearity that $u_p'' + \frac{1}{4}u_p =$ the Fourier series of $f(x)$ which equals $f(x)$, almost everywhere. We can rewrite u_p as

$$u_p = \sum_{k=1}^{\infty} \frac{\frac{4}{(2k-1)\pi} \sin((2k-1)x)}{\frac{1}{4} - (2k-1)^2} = \frac{16}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)x)}{(2k-1)(1 - 4(2k-1)^2)}.$$

Now, we have that the general solution is

$$u = u_h + u_p = C_1 \sin(x/2) + C_2 \cos(x/2) + u_p.$$

We still need to find C_1 and C_2 such that $u(0) = u'(0) = 0$.

$$u(0) = 0 + C_2 + \frac{16}{\pi} \sum_{k=1}^{\infty} 0 = C_2.$$

Thus $C_2 = 0$.

Now,

$$u'(x) = \frac{C_1}{2}\cos(x/2) + \frac{16}{\pi}\sum_{k=1}^{\infty}\frac{(2k-1)\cos((2k-1)x)}{(2k-1)(1-4(2k-1)^2)}.$$

Thus,

$$u'(0) = \frac{C_1}{2} + \frac{16}{\pi}\sum_{k=1}^{\infty}\frac{1}{1-4(2k-1)^2}.$$

It turns out $\sum_{k=1}^{\infty}\frac{1}{1-4(2k-1)^2} = -\pi/8$. (I'll show how to do this in a bit.) Then, since $u'(0) = 0$ we have

$$C_1 = \frac{32}{\pi}\cdot\frac{\pi}{8} = 4.$$

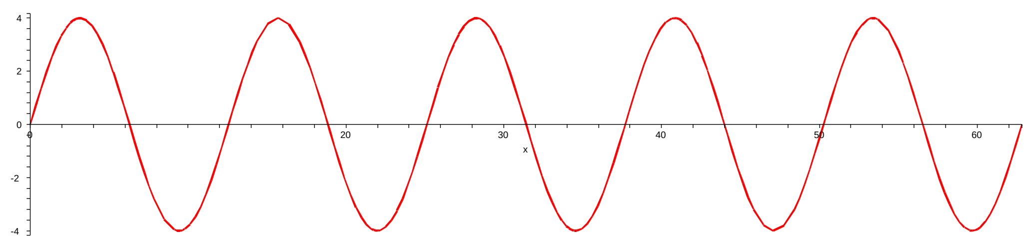
Our final solution is

$$u(x) = 4\sin(x/2) + \frac{16}{\pi}\sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{(2k-1)(1-4(2k-1)^2)}.$$

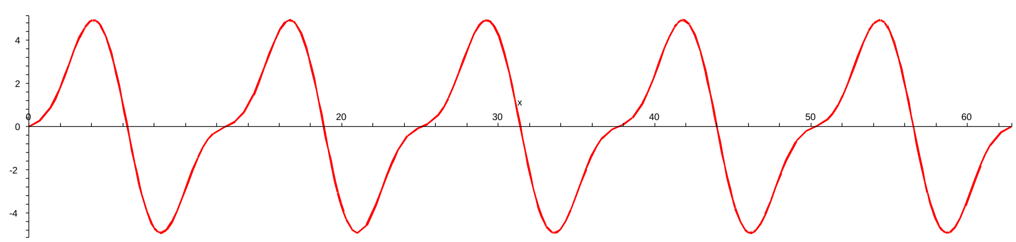
Next we plot some partial sums.

11.1 Graphs for ODE Example

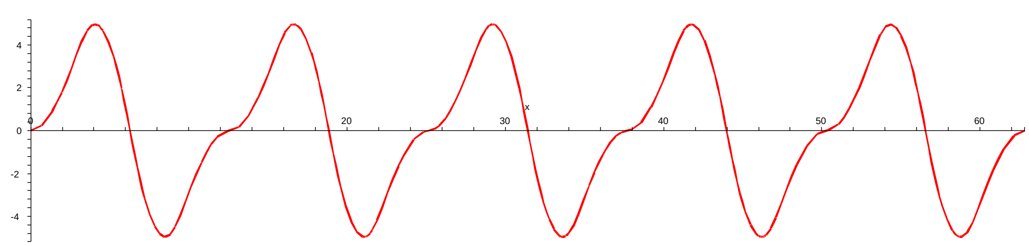
$4\sin x/2$



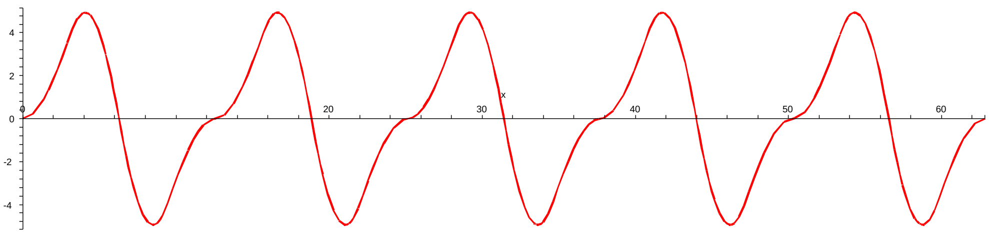
$N = 1$



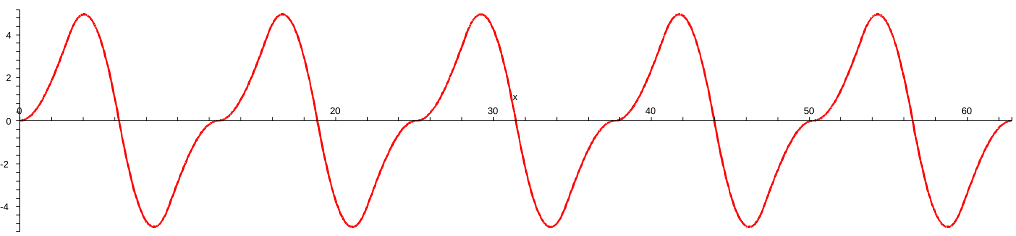
$N = 2$



$N = 3$



$N = 10$



11.2 Back to computing that series...

Back to why $\sum_{k=1}^{\infty} \frac{1}{1-4(2k-1)^2} = -\pi/8$.

Method one. Use a computer. Here is the command in Maple:

$$\texttt{sum}(1/(4*(2*\texttt{k}-1)^2-1), \texttt{k}=1..\texttt{infinity});$$

It returns $\pi/8$. (I switched the terms in the denominator to make it positive.) I could not find a way to do this is Maxima; it can only do numerical approximations.

Method two. Recall that you took a course called *Calculus II*. Look up the Taylor series for $\arctan x$, then plug in $x = 1$. This shows that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \arctan(1) = \frac{\pi}{4}.$$

We will apply this fact to our sum. But, first we notice that

$$\begin{aligned} \frac{1}{4(2k-1)^2-1} &= \frac{1}{16k^2-16k+3} \\ &= \frac{1}{(4k-3)(4k-1)} \\ &= \frac{\frac{1}{2}}{4k-3} - \frac{\frac{1}{2}}{4k-1}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{4(2k-1)^2-1} &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{4k-3}-\frac{1}{4k-1}\right) \\
&= \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right) \\
&= \frac{1}{2}\frac{\pi}{4}=\frac{\pi}{8}.
\end{aligned}$$

Note: There is a subtle issue with convergence here. The radius of convergence of the Taylor series of $\arctan x$ is 1. At $x = 1$ the series does converge by the alternating series test. But the proof that it converges to $\arctan 1$ requires the Weierstrass M-test which is covered in Math 352.

12 The Wave Equation for a Vibrating String

We are going to use Fourier series to solve some partial differential equations. The methods used will seem quite strange and ad hoc at first. They are in fact standard and are used to solve a number of important PDE's used in physics, chemistry and engineering.

We shall start will a classic problem of modeling a vibrating string. When a string a held straight and not moving it is in equilibrium. We will start the motion but plucking it, that is we pinch a straight string in the middle and pull in up a bit a let it go.

Let the string have length $L > 0$ and use $x \in [0, L]$ as a as the distance to the left end point. Let t be time. Let $u(x, t)$ be the height of the string away from equilibrium. If $u(x, t)$ is negative then that point of the string is below equilibrium.

The PDE that is used to model this is called **the wave equation**. It is

$$\frac{\partial^2 u}{\partial x^2} = k\frac{\partial^2 u}{\partial t^2},$$

where $k > 0$ is a constant. This rationale is that the force at any point should be proportional to the concavity at that point. It is a simplified model; for example it does not include a term for air resistance or internal heat loss. Real strings can wear out and break, but not our ideal string. It is customary to rewrite the wave equation as

$$a^2 u_{xx} = u_{tt},$$

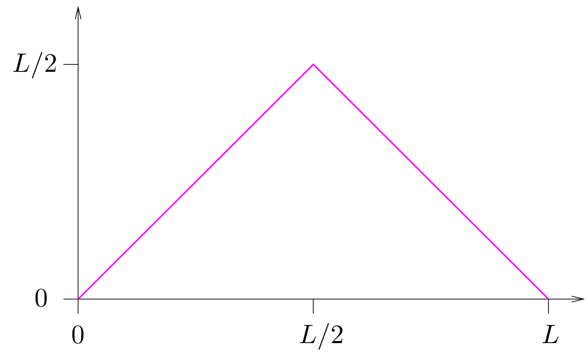
where $a > 0$ is a constant.

Next we add **boundary conditions**. These simply model the fact that we shall be holding the end points fixed. Thus, for all t,

$$u(0, t) = u(L, t) = 0.$$

Finally we add the **initial conditions** or **configuration**. For simplicity we shall give use $u(L/2, 0) = L/2$ and assume the string is piecewise linear. Then

$$u(x, 0) = f(x) = \begin{cases} x & \text{for } x \in [0, L/2], \\ L - x & \text{for } x \in (L/2, L]. \end{cases}$$



We encode the fact that initially the string is not moving by

$$u_t(x,0) = 0$$

for all $x \in [0, L]$. In general we could specify $u_t(x,0) = g(x)$ for some function $g(x)$.

Thus, our model is as shown in the box below.

$$a^2 u_{xx} = u_{tt}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = 0$$

where $f(x)$ was given above.

12.1 Solving the Model

Our strategy is as follows. Suppose $u(x,t)$ can be written in the form

$$u(x,t) = X(x)T(t).$$

Plugging into the wave equation we get

$$a^2 X''(x)T(t) = X(x)T''(t).$$

Now we separate the terms involving x from the terms involving t to get

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)}.$$

Both x and t are free, independent variables. But, if we change x notice the right hand side will not change because it depends only on t. Therefore $X''/X =$ a constant. Likewise $T''/a^2T =$ the same constant. We will call this constant $-\sigma$ because we can. Thus, we now have two second order ODEs:

$$X'' + \sigma X = 0 \qquad \& \qquad T'' + \sigma a^2 T = 0.$$

For the first we can deduce boundary conditions.

$$u(0,t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0.$$

Likewise $X(L) = 0$. So, let's just focus on the problem

$$X''(x) + \sigma X(x) = 0 \qquad X(0) = X(L) = 0.$$

We know that the form of the solution will depend on the sign of σ. We will consider three cases, $\sigma < 0$, $\sigma = 0$ and $\sigma > 0$, and see what happens. It will turn out that nontrivial solutions will exist only for certain values of σ.

Case I. Suppose $\sigma < 0$. Then the general solution is

$$X(x) = C_1 e^{\sqrt{-\sigma}x} + C_2 e^{-\sqrt{-\sigma}x}.$$

We impose the boundary conditions $X(0) = X(L) = 0$ to get C_1 and C_2.

We get that $X(0) = 0$ implies $C_1 + C_2 = 0$ and $C_1 e^{\sqrt{-\sigma}L} + C_2 e^{-\sqrt{-\sigma}L} = 0$. Thus,

$$C_1 = -C_2$$

and

$$C_1 = -e^{-2\sqrt{-\sigma}L} C_2.$$

Thus we can only get a solution besides $C_1 = C_2 = 0$ if $e^{-2\sqrt{-\sigma}L} = 1$. But, this implies $\sigma = 0$. We conclude that there are no nontrivial solutions if $\sigma < 0$.

Case II. Suppose $\sigma = 0$. Now our equation is just $X''(x) = 0$. The general solution is

$$X(x) = C_1 x + C_2.$$

Notice, $X(0) = 0$ implies $C_2 = 0$, but then $X(L) = 0$ implies $C_1 L = 0$ giving us that $C_1 = 0$. Again, there are no nontrivial solutions when $\sigma = 0$. It seems we are wasting our time!

Case III. As a final act of desperation suppose $\sigma > 0$. Now the general solution is

$$X(x) = C_1 \sin \sqrt{\sigma}x + C_2 \cos \sqrt{\sigma}x.$$

Now, $X(0) = C_2$, which implies $C_2 = 0$. But $X(L) = C_1 \sin(\sqrt{\sigma}L)$. This implies $C_1 = 0$ unless the sine of $\sqrt{\sigma}L$ is zero. So, our only hope of finding nontrivial solutions is to suppose $\sqrt{\sigma}L$ is an integer multiple of π. That is

$$\sigma = \frac{n^2\pi^2}{L^2}$$

for some positive integer n. For these values

$$X(L) = C_1 \sin\left(\sqrt{\sigma}L\right) = C_1 \sin(n\pi) = 0,$$

for any value of C_1! Thus, for any nonzero integer n we have infinitely many nontrivial solutions; we will take n to be positive.

This is not the behavior you are used from working with initial value problems for ODEs. There we got unique solutions. But, so far we have only done the *boundary values* and we have not yet considered the initial shape and velocity on our string. We will record our observation of nontrivial solutions by defining

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \text{ for } n \geq 1.$$

Next we study the $T(t)$ equation.

Recall

$$T''(t) + a^2\sigma T(t) = 0.$$

Since we can only get nontrivial solutions for $X(x)$ if $\sigma = \dfrac{n^2\pi^2}{L^2}$ we shall assign this value to σ. Thus, we have

$$T''(t) + \frac{a^2n^2\pi^2}{L^2}T(t) = 0.$$

The general solution is

$$T(t) = C_1 \sin\left(\frac{an\pi t}{L}\right) + C_2 \cos\left(\frac{an\pi t}{L}\right).$$

Now we shall begin looking at the initial conditions. It turns out it is easier if we study the initial velocity first. We were given

$$u_t(x,0) = 0,$$

meaning that the string is not moving at the moment we let go of it. Now $u_t = \partial_t(X(x)T(t)) = X(x)T'(t)$. Thus we have

$$X(x)T'(0) = 0.$$

Assuming we have a nontrivial solution for $X(x)$ this forces $T'(0) = 0$. We compute

$$T'(t) = C_1 \frac{an\pi}{L} \cos\left(\frac{an\pi t}{L}\right) - C_2 \frac{an\pi}{L} \sin\left(\frac{an\pi t}{L}\right).$$

Thus,

$$T'(0) = C_1 \frac{an\pi}{L} = 0 \implies C_1 = 0.$$

There are no restrictions on C_2. For any value of C_2

$$T(t) = C_2 \cos\left(\frac{an\pi t}{L}\right)$$

solves the differential equation in T and the condition $T'(0) = 0$.

We let $T_n(t) = \cos\left(\frac{an\pi t}{L}\right)$ and define $u_n(x,t) = X_n(x)T_n(t)$.

Each u_n satisfies the wave equation, the two boundary conditions, and the initial velocity condition. Furthermore, by linearity, any linear combination of the u_n's will also satisfy these conditions. **Check this.**

It can be shown that if we let

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

and the c_n's are such that it converges everywhere, then u satisfies the wave equation, the two boundary conditions, and the initial velocity condition. This in proven in Math 407.

The final step, is that we need to choose the c_n's so that

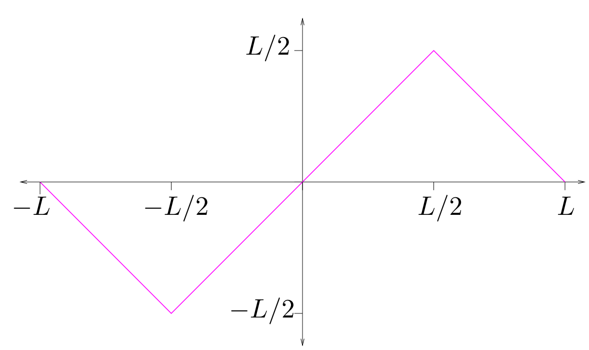
$$\sum_{n=1}^{\infty} c_n u_n(x,0) = f(x).$$

We do this next.

We need for

$$f(x) = \sum_{n=1}^{\infty} c_n u_n(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi 0}{L}\right) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right).$$

But, this is just the Fourier series for the odd periodic extension of $f(x)$.



Thus,

$$\begin{aligned}
c_n &= \frac{1}{L}\int_{-L}^{L} \tilde{f}_o(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{2}{L}\int_{0}^{L/2} x\sin\left(\frac{n\pi x}{L}\right)\,dx + \frac{2}{L}\int_{L/2}^{L}(-x+L)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \text{..... busy work ....} \\
&= \frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi t}{L}\right).$$

We can rewrite this as

$$u(x,t) = \frac{4L}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\left(\frac{(2k+1)\pi x}{L}\right)\cos\left(\frac{a(2k+1)\pi t}{L}\right).$$

12.2 Graphs and an Animation

Now for the fun part. We will make an animation of $u(x,t)$. We shall take $a = 1$ and $L = 2$.

```
> with(plots);                              # Load package of plotting commands.
> c:= n -> 8*sin(n*Pi/2)/(n^2*Pi^2):        # Define the coefficient function.
> N:=100:                                   # Number of terms to use.
>                                           # Finally we execute the animation command.
> animate({sum((c(n)*sin(n*Pi*x/2)*cos(n*Pi*t/2)), n=1..N)},
    x=0..2, t=0..20, frames=150, thickness=2);
```

See the animation link on the course web site.

12.3 Summary for Wave Equation

Our model for a vibrating string is repeated below.

$$a^2u_{xx} = u_{tt}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x) \qquad u_t(x,0) = 0$$

where $f(x)$ is given.

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{an\pi t}{L}\right)$$

where

$$c_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx.$$

12.4 Extra Credit!

Modify the wave equation by adding a damping term, $\gamma \dfrac{\partial u}{\partial t}$. Use the same boundary and initial conditions as in the example we just did. Solve this model. Write this up neatly and turn it in. Make an animation, put it on web site, and send me the link. When you are at an interview for a job or internship, bring your tablet or phone, and show your animation to the interviewer.

13 The Heat Equation

We consider a metal rod of length L. Suppose that initially the temperature at each point $x \in [0, L]$ along the rod is given by $f(x)$. We wish to know how the temperature will evolve over time. Let

$$u(x,t)$$

be the temperature at location x and time t. Thus, $u(x,0) = f(x)$. The evolution of the temperature along to rod is modeled by the **heat equation**, which is

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

or $\alpha^2 u_{xx} = u_t$. See Appendix in textbook.

We will make an additional assumption that temperature of the ends of the metal rod are held at fixed temperatures. This gives the **boundary conditions**

$$u(0,t) = T_1 \qquad \& \qquad u(L,t) = T_2.$$

Later we will consider other types of boundary conditions.

We will solve the present problem in two steps. First, we assume $T_1 = T_2 = 0$. Then we will show how to jazz this up to the more general case.

13.1 Solving Homogeneous Case

Step 1 is to solve the following model.

$$\alpha^2 u_{xx} = u_t$$
$$u(0,t) = u(L,t) = 0$$
$$u(x,0) = f(x)$$

where $f(x)$ will be given.

Solution. Suppose $u(x,t) = X(x)T(t)$. Then $u_{xx}(x,t) = X''(x)T(t)$ and $u_t(x,t) = X(x)T'(t)$. Therefore,

$$\alpha^2 X''T = XT' \qquad \Longrightarrow \qquad \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}.$$

This implies that each side of the second equation above must be constant. We call the constant $-\sigma$. This yields two ordinary differential equations.

$$X'' + \sigma X = 0 \qquad \& \qquad T' + \sigma\alpha^2 T = 0.$$

The boundary conditions $u(0,t) = u(L,t) = 0$ implies $X(0) = X(L) = 0$. Then for the equation in X the analysis we did for the wave equation goes through in exactly the same way. We conclude that non trivial solution exist only when $\sqrt{\sigma} = \dfrac{n\pi}{L}$ where n is an integer, and that

$$X(x) = C \sin\left(\frac{n\pi x}{L}\right)$$

gives a solution for any real number C and integer n. Let

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \text{for} \quad n \geq 1.$$

Note: $n = 0$ gives a trivial solution and negative values for n are redundant.

Now, consider the ordinary differential equation in T. Its general solution is just

$$T(t) = Ce^{-\sigma\alpha^2 t} = Ce^{-\frac{n^2\pi^2\alpha^2}{L^2}t}.$$

We let

$$T_n(t) = e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \qquad \& \qquad u_n(x,t) = X_n(x)T_n(t) \quad \text{for} \quad n \geq 1.$$

We have

$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \quad \text{for } n \geq 1.$$

Each u_n satisfies the heat equation and the boundary conditions. So would any linear combination of u_n. It is shown in Math 407 that if $\{c_n\}_{n=1}^{\infty}$ are such that

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

converges, then $u(x,t)$ satisfies the heat equation (almost everywhere) and the boundary conditions.

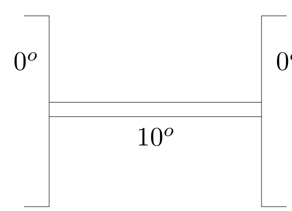
Therefore, if we can find values for the c_n's such that

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

where $f(x)$ is the initial temperature distribution, then $u(x,t)$ will satisfy all our requirements.

13.2 An Example: Metal Rod with Ends held at Zero Degrees

Suppose $f(x) = 10$, $L = 1$ and $\alpha = 1$. At the instant $t = 0$ the end points of the rod put into contact with degree zero heat sinks. We will model what happens next.



The odd periodic extension of $f(x)$ is a square wave. Let $L = 1$. We compute the c_n's.

$$c_n = \frac{1}{1}\int_{-1}^{1} \sin(n\pi x) f_o(x)\,dx = 20\int_0^1 \sin(n\pi x)\,dx = -\frac{20}{n\pi}\big[\cos(n\pi) - \cos(0)\big] = \begin{cases} \dfrac{40}{n\pi} & \text{for} \quad n \text{ odd,} \\ 0 & \text{for} \quad n \text{ even.} \end{cases}$$

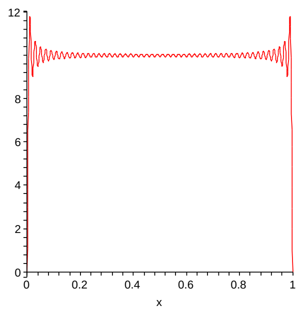
Thus,

$$u(x,t) = \sum_{k=1}^{\infty} \frac{40}{(2k-1)\pi} \sin((2k-1)\pi x) e^{-(2k-1)^2\pi^2 t}.$$

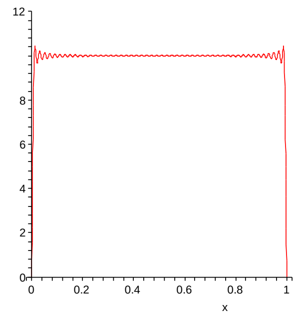
An animation is on the web page. Below are some snapshots.

Some Plots for the Last Example

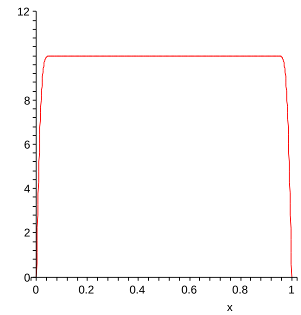
$t = 0.0$



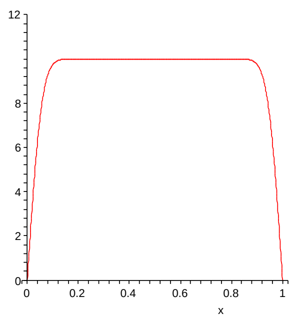
$t = 0.00001$



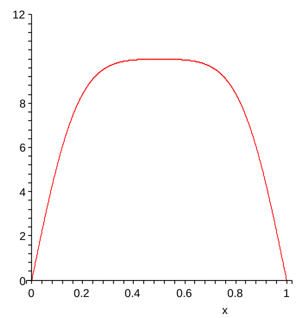
$t = 0.0001$



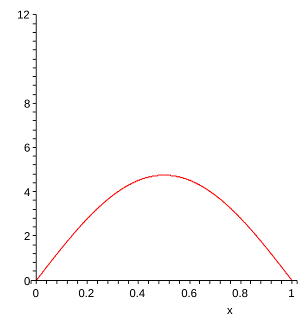
$t = 0.001$



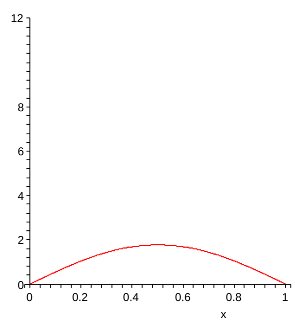
$t = 0.01$



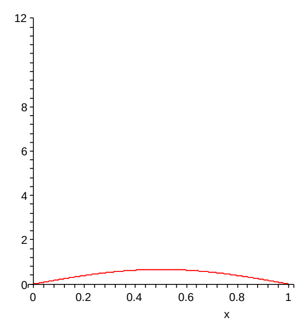
$t = 0.1$



$t = 0.2$



$t = 0.3$

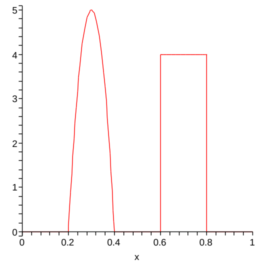


13.3 A Crazy Example

Suppose $L = 1$, $\alpha = 1$ and the initial temperature distribution is given by

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 \le x < 0.2 \\ -500(x-0.2)(x-0.4) & \text{for} \quad 0.2 \le x < 0.4 \\ 0 & \text{for} \quad 0.4 \le x < 0.6 \\ 4 & \text{for} \quad 0.6 \le x < 0.8 \\ 0 & \text{for} \quad 0.8 \le x < 1.0 \end{cases}$$

Below is a graph.



Here are the Maple commands that created it.

```
> f := x -> piecewise(x<.2, 0, x>=.2 and x<=.4 , -500*(x-0.2)*(x-0.4),
x>0.4 and x<0.6, 0, x>=0.6 and x<=0.8, 4, x>0.8, 0);
> plot(f(x),x=0..1);
```

The coefficients are given by $c_n = 2\int_0^1 \sin(n\pi x) f(x)\,dx$. I did not bother trying to find a formula for the c_n, I just used the integral. Thus the solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) e^{-n^2\pi^2 t}.$$

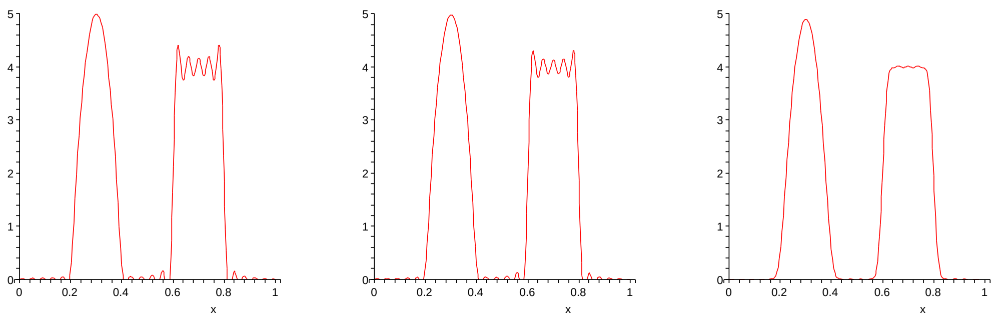
I created an animation using the commands below. I took the sum out to $n = 50$, but this is likely overkill. It is on the web-page.

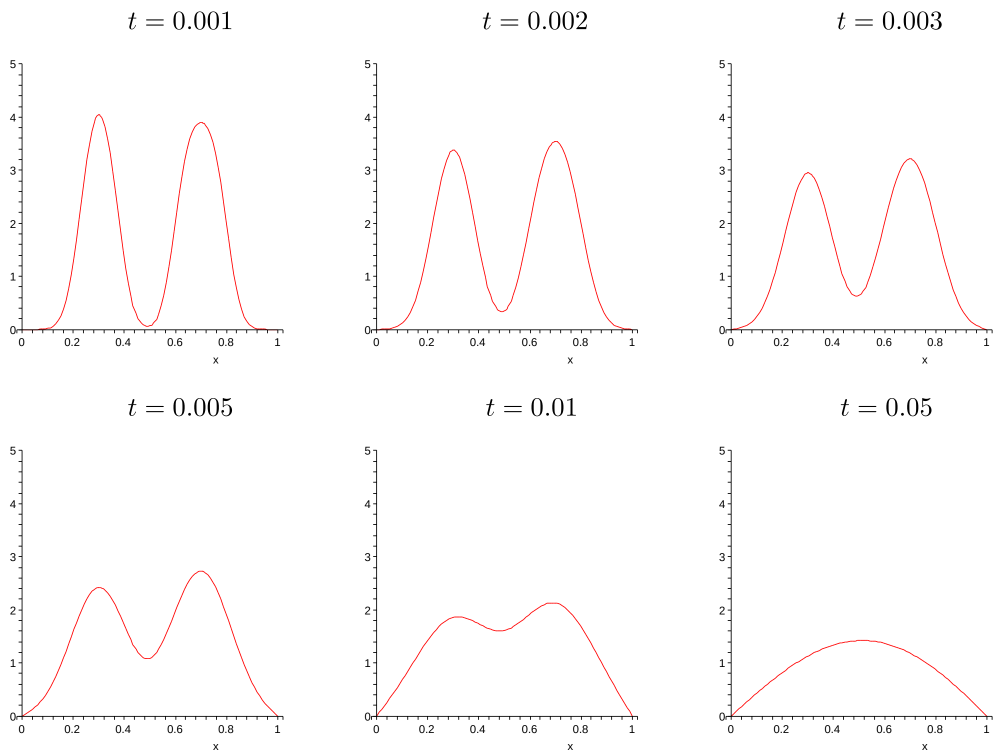
```
> c := n -> 2*int(sin(n*Pi*x)*f(x) ,x=0..1);
> animate(sum( c(n)*sin((n)*Pi*x)*exp(-(n^2*Pi^2*t)) ,n=1..50),
x=0..1,t=0..0.2,numpoints=1000,frames=100);
```

Snapshots of the animation are below.

Some Plots for the Last Example

$t = 0.0$ $\qquad$ $t = 0.00001$ $\qquad$ $t = 0.0001$





14 Heat Equation with Nonhomogeneous Boundary Conditions

Now we are working with the model

$$\alpha^2 u_{xx} = u_t$$

$$u(0,t) = T_1 \qquad \& \qquad u(L,t) = T_2$$

$$u(x,0) = f(x)$$

where $f(x)$ will be given.

Key Idea. As $t \to \infty$ we think that $u(x,t) \to \frac{T_2 - T_1}{L}x + T_1$. So, we let

$$v(x) = \frac{T_2 - T_1}{L}x + T_1 \qquad \text{and} \qquad w(x,t) = u(x,t) - v(x).$$

Notice that $w_{xx} = u_{xx}$ and $w_t = u_t$. Therefore, w satisfies the heat equation

$$\alpha^2 w_{xx} = w_t.$$

Further, $w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0$ and likewise $w(L,t) = 0$. However,

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x).$$

Let $g(x) = f(x) - v(x)$. Then $w(x,t)$ satisfies the model below.

$$\alpha^2 w_{xx} = w_t$$

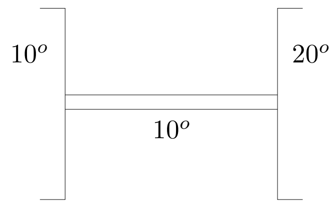
$$w(0,t) = w(L,t) = 0$$

$$w(x,0) = g(x).$$

Our plan is to solve this model for $w(x,t)$ and then add $v(x)$ to get $u(x,t)$. We do this in the next example.

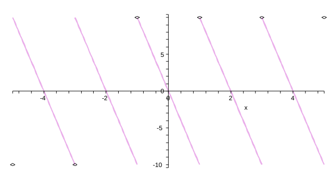
14.1 Example: Metal Rod Ends Held at Different Temperatures

Let $L = 1$, $T_1 = 10^o$ and $T_2 = 20^o$. We suppose the bar starts with uniform temperature 10^o and at time $t = 0$ the ends are set to a heat bath of T_1 or the right and T_2 on the left.



Thus, $v(x) = 10x + 10$ and $g(x) = f(x) - v(x) = 10 - (10x + 10) = -10x$. The graph of the odd periodic extension of $g(x)$ is below along with the command that created it. Note: `frac` is a Maple command that gives the fractional part of a number.

```
> plot(piecewise(x>=-1,-20*frac((x+1)/2)+10,x<-1,-20*frac((x+1)/2)-10),
x=-5..5,discont=true, color=plum, thickness=2);
```



Now we compute the Fourier series coefficients for the odd periodic extension of $g(x)$.

$$c_n = \frac{1}{1}\int_{-1}^{1} \sin(n\pi x) g_o(x)\, dx = -20\int_0^1 x\sin(n\pi x)\, dx = \frac{20}{n\pi}\cos(n\pi) = \frac{20}{n\pi}(-1)^n.$$

Thus,

$$w(x,t) = \frac{20}{\pi}\sum_{n=1}^{\infty} \tfrac{(-1)^n}{n} \sin(n\pi x) e^{-n^2\pi^2 t}.$$

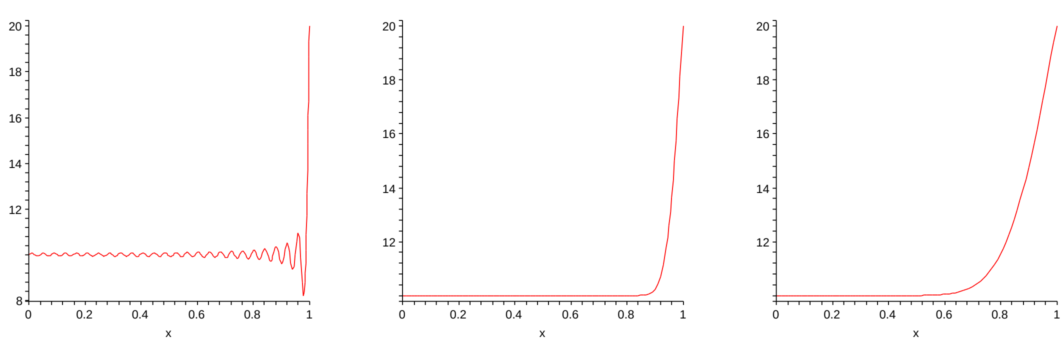
Hence,

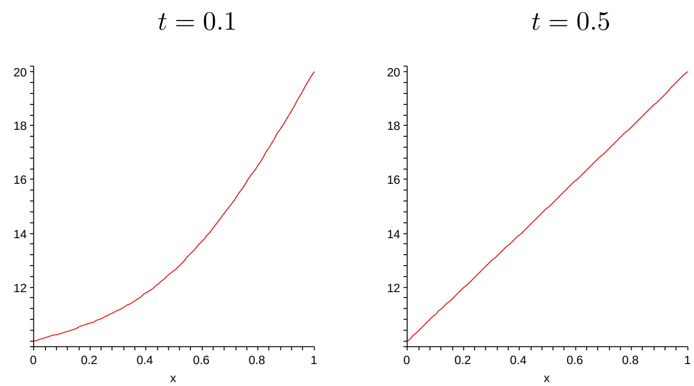
$$u(x,t) = v(x) + w(x,t) = 10 + 10x + \frac{20}{\pi}\sum_{n=1}^{\infty} \tfrac{(-1)^n}{n} \sin(n\pi x) e^{-n^2\pi^2 t}.$$

Some plots are below. Animation coming soon to a website near you!

Plots for Last Example

$t = 0.0$ $\qquad$ $t = 0.001$ $\qquad$ $t = 0.01$





15 Heat Equation with Insulated Ends

Now we consider a metal rod where heat cannot flow out through the ends. The end are insulated. The model now becomes the following.

$$\alpha^2 u_{xx} = u_t$$
$$u_x(0,t) = u_x(L,t) = 0$$
$$u(x,0) = f(x),$$
where $f(x)$ is given.

We go through the same steps are before. Suppose $u(x,t) = X(x)T(t)$. Then the heat equation becomes

$$\alpha^2 X''(x)T(t) = X(x)T'(t),$$

and just as before we have

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = \text{ a constant } = -\sigma.$$

Thus, we get two ODE's

$$X'' + \sigma X = 0 \qquad \& \qquad T' + \sigma\alpha^2 T = 0.$$

We work with the X equation first. We consider the same three cases $\sigma < 0$, $\sigma = 0$ and $\sigma > 0$. The boundary condition translates to

$$X'(0) = X'(L) = 0.$$

Suppose σ<0. It will turn out there are no nontrivial solutions. The general solution is

$$X(x) = C_1 e^{\sqrt{-\sigma}x} + C_2 e^{-\sqrt{-\sigma}x}$$

Then

$$X'(x) = C_1\sqrt{-\sigma}e^{\sqrt{-\sigma}x} - C_2\sqrt{-\sigma}e^{-\sqrt{-\sigma}x}$$

At the boundary points $x = 0$ and $x = L$ we have

$$X'(0) = C_1\sqrt{-\sigma} - C_2\sqrt{-\sigma} = 0 \quad \Longrightarrow \quad C_1 = C_2$$

and

$$X'(L) = C_1\sqrt{-\sigma}e^{\sqrt{-\sigma}L} - C_2\sqrt{-\sigma}e^{-\sqrt{-\sigma}L} \quad \Longrightarrow \quad C_1 = C_2 e^{-2\sqrt{-\sigma}L}$$

These only have a solution besides $C_1 = C_2 = 0$ if $L = 0$, which we do not allow.

Suppose σ=0. The general solution is $X(x) = C_1 x + C_2$. Since $X'(x) = C_1$ it must be that $C_1 = 0$. Any $X(x) = C_2$ will be a solution.

Finally, suppose σ>0. The general solution is $X(x) = C_1 \sin\sqrt{\sigma}x + C_2 \cos\sqrt{\sigma}x$. Thus,

$$X'(x) = C_1\sqrt{\sigma}\cos\sqrt{\sigma}x - C_2\sqrt{\sigma}\sin\sqrt{\sigma}x.$$

Thus,

$$X'(0) = C_1\sqrt{\sigma} = 0 \quad \Longrightarrow \; C_1 = 0.$$

Then

$$X'(L) = -C_2\sqrt{\sigma}\sin\sqrt{\sigma}L. = 0 \quad \Longrightarrow \quad \sigma = \frac{n^2\pi^2}{L^2} \text{ or } C_2 = 0.$$

Thus, if we want nontrivial solutions we require $\sigma = \frac{n^2\pi^2}{L^2}$. Then there are no restrictions on C_2. We let

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right) \quad \text{for } n = 0, 1, 2, 3, ...$$

Notice $n = 0$ gives us a constant.

Recall the equation for $T(t)$ was $T' + \sigma\alpha^2 T = 0$. It is general solution is

$$T(t) = Ce^{-\sigma\alpha^2 t}.$$

We let

$$T_n(t) = e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \quad \text{for } n = 0, 1, 2, 3, ...$$

Let

$$u_n(x,t) = X_n(x)T_n(t) \;\; : \quad \text{for } n = 0, 1, 2, 3, ...$$

Each $u_n(x,t)$ satisfies the heat equation and the boundary conditions on u_x. So too does any linear combination of them. In is shown in Math 407 that if $c_0, c_1, c_2, c_3, \ldots$ are such that

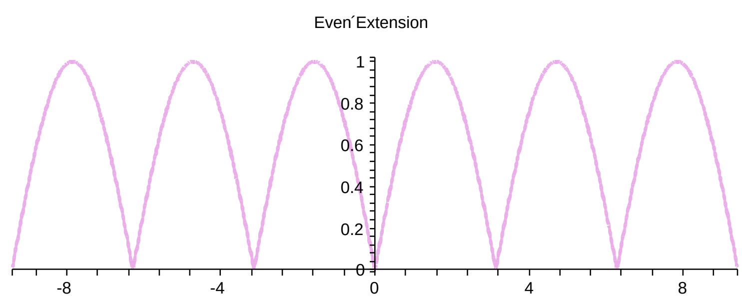
$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n X_n T_n$$

converges, then $u(x,t)$ satisfies the heat equation and the boundary conditions on u_x. All that is left to do is find c_n's such that $u(x,0) = f(x)$, the initial condition. But this is just the Fourier series of the **even** periodic extension of $f(x)$. Thus, c_0 is the average value of $f(x)$ and for $n = 1, 2, 3, ...$

$$c_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f_e(x)\,dx = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx.$$

15.1 Example for Heat Equation with Insulated Ends

Suppose $L = \pi$, $\alpha = 1$ and $u(x,0) = f(x) = \sin(x)$. The even periodic extension of $f(x)$ is $|\sin(x)|$.



But, using symmetry our integrals are only over $[0, \pi]$. Then

$$c_0 = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}$$

and for $n = 1, 2, 3, \ldots$ we have

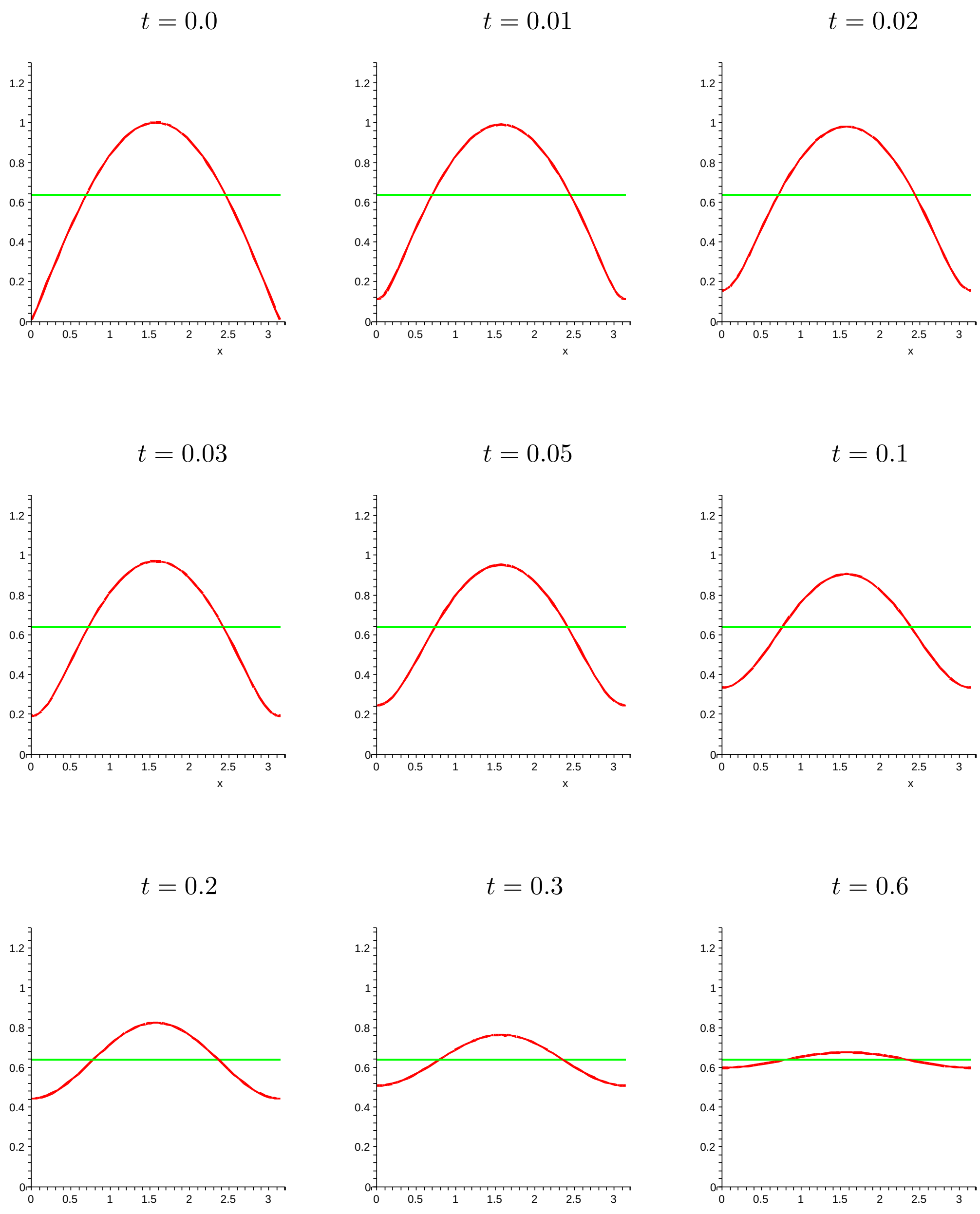
$$c_n = \frac{2}{\pi}\int_0^{\pi} \cos\left(\frac{n\pi x}{\pi}\right)\sin(x) = \frac{1+\cos(n\pi)}{1-n^2} = \begin{cases} \frac{4/\pi}{1-n^2} & \text{for} \quad n \text{ even,} \\ 0 & \text{for} \quad n \text{ odd.} \end{cases}$$

Note, the $n = 1$ case should done separately to avoid division by zero, but it comes out zero. Therefore,

$$u(x,t) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{1-4k^2}\cos(2kx)e^{-4k^2 t}.$$

On the next page are plots for various values of t with the sum going to $k = 30$. The green lines mark $2/\pi$. An animation is on the course website.

Plots for Last Example



16 Summary for Heat Equation

End points held at 0 degrees (homogeneous case).

Model: $\alpha^2 u_{xx} = u_t$, $u(0,t) = u(L,t) = 0$, $u(x,0) = f(x)$.

Solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}}$$

where

$$c_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx.$$

End points held at fixed temperatures (Nonhomogeneous case).

Model: $\alpha^2 u_{xx} = u_t$, $u(0,t) = T_1$, $u(L,t) = T_2$, $u(x,0) = f(x)$.

Solution: Let $v(x) = \frac{T_2 - T_1}{L}x + T_1$. Then

$$u(x,t) = v(x) + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}},$$

where

$$c_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) [f(x) - v(x)]\,dx.$$

Insulated end points.

Model: $\alpha^2 u_{xx} = u_t$, $u_x(0,t) = 0$, $u_x(L,t) = 0$, $u(x,0) = f(x)$.

Solution:

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha^2 t}{L^2}}$$

where

$$c_0 = \text{ the average value of } f(x) \;= \frac{1}{L}\int_0^L f(x)\,dx$$

and for $n = 1, 2, 3, \ldots$

$$c_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx.$$

17 Extra Credit!

Consider a metal ring. For a given initial temperature distribution $f(\theta)$ for $\theta \in [0, 2\pi]$ set up a model with the heat equation and show how to solve it. Then work through an example and display the plots for several times. If you like, create an animation.

18 The Heat Equation for a Square Plate: OPTIONAL READING

Let $u(x,y,t)$ be the temperature at (x,y) at time t. The two dimensional heat equation in rectangular coordinates is

$$\alpha^2(u_{xx}+u_{yy})=u_t.$$

For simplicity we will set $\alpha=1$, work on the 1×1 square with corners (0,0), (1,0), (1,1), (0,1), and for a boundary condition we suppose the boundary is held to 0 degrees.

Suppose $u(x,y,t)=X(x)Y(y)T(t)$. Then the heat equation becomes

$$X''YT+XY''T=XYT'.$$

Divide through by XYT to get

$$\frac{X''Y+XY''}{XY}=\frac{T'}{T}$$

Since the left does not change with t, both sides are equal to some constant; call it $-\sigma$. Now we have

$$T'+\sigma T=0 \;\;\&\;\; \frac{X''Y+XY''}{XY}=-\sigma.$$

The equation in T we know how to deal with. For the equation in X and Y we have

$$\frac{X''}{X}=-\sigma-\frac{Y''}{Y}.$$

Since one side depends only on x and the other only on y, both must be a constant; call it $-\beta$. Now we have

$$X''+\beta X=0 \;\;\&\;\; Y''+(\sigma-\beta)Y=0.$$

The equation in X will only have nontrivial solutions if $\beta=n^2\pi^2$ for some integer $n>0$. You can show that $X(x)$ can then be any multiple of $\sin(n\pi x)$. We let

$$X_n(x)=\sin(n\pi x).$$

The equation in Y will only have nontrivial solutions if $\sigma-\beta=m^2\pi^2$ for some integer $m>0$. Thus $\sigma=(n^2+m^2)\pi^2$. We let

$$Y_m(y)=\sin(m\pi y),$$
$$T_{m,n}(t)=e^{-(m^2+n^2)\pi^2 t},$$

and

$$u_{m,n}(x,y,t)=X_n(x)Y_m(y)T_{m,n}(t).$$

Then $u_{m,n}$ satisfies the heat equation and the boundary condition for any integers $n>0$, $m>0$ as would any linear combination.

Now, the theory of Fourier series can be extended to functions of two variables. If $f(x,y)$ is odd in both variables and periodic with period $2L$ is each variable then the series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{m,n}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)$$

where

$$c_{m,n}=\frac{4}{L}\int_0^L\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)f(x,y)\,dx\,dy$$

converges, almost everywhere, to $f(x,y)$. [See *Fourier Series and Boundary Value Problems* by Brown and Churchill.]

Thus, if we are given $u(x,y,0)=f(x,y)$ we just use the above formulas for the $c_{m,n}$. Then

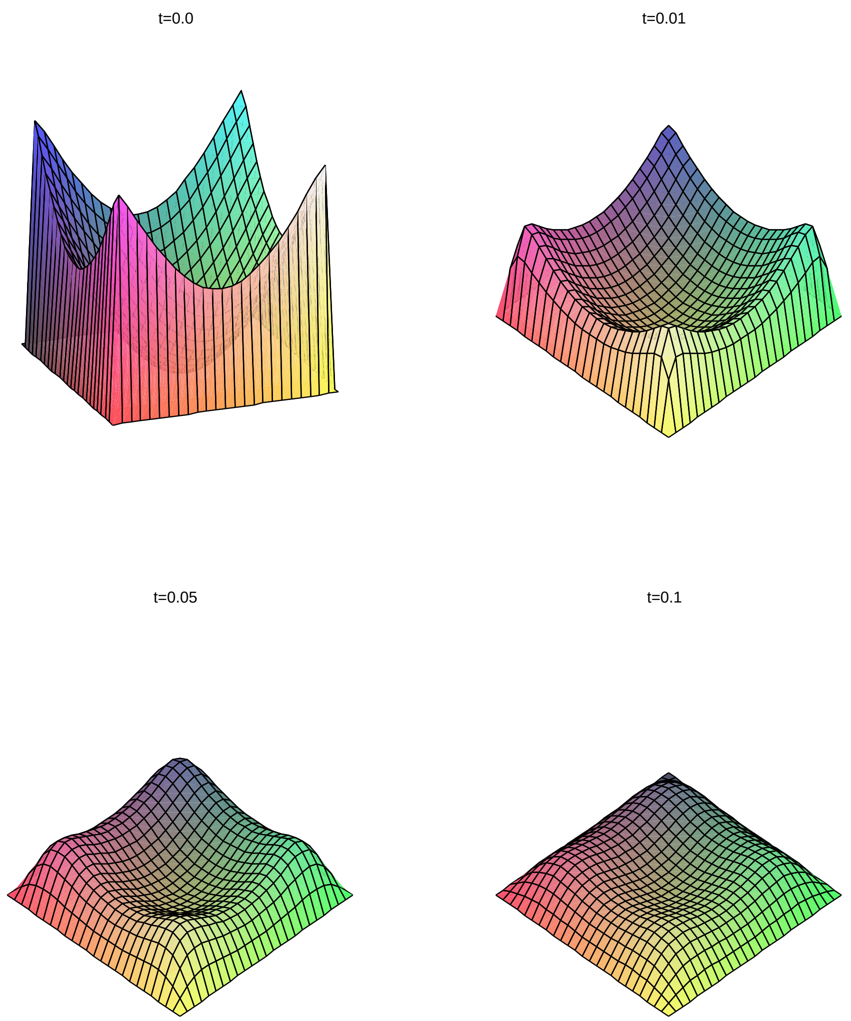
$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{m,n}\sin(n\pi x)\sin(m\pi y)e^{-(m^2+n^2)\pi^2 t}.$$

18.1 Example

Suppose $u(x, y, 0) = f(x, y) = (x - 0.5)^2 + (y - 0.5)^2$. Then we can compute the $c_{m,n}$'s and animate the result by the commands below.

```
> c := (m,n) -> 4*int(int(sin(n*Pi*x) * sin(m*Pi*y) *
((x-0.5)^2 + (y-0.5)^2),x=0..1),y=0..1);

> animate(plot3d ,[sum(sum(c(m,n)*sin(n*Pi*x)*sin(m*Pi*y)*
exp(-(m^2+n^2)*t),m=1..10),n=1..10),x=0..1,y=0..1],t=0..0.3,frames=100);
```



See the course website to view the animation.

19 Heat Equation on a Metal Disk: OPTIONAL READING

We consider a metal disk of radius 1 with temperature held to 0 degrees around the circumference. The initial temperature is given by $f(r, \theta)$. Let $u(r, \theta, t)$ be the temperature at location (r, θ) at time t. Thus,

$$u(1, \theta, t) = 0 \quad \& \quad u(r, \theta, 0) = f(r, \theta).$$

Now, the heat equation in rectangular coordinates is

$$\alpha^2(u_{xx} + u_{yy}) = u_t.$$

Our first task is to translate this into polar coordinates. I'll show one of the steps involved and then give the final result. The main tool is the multivariable chain rule.

$$\frac{\partial u(r,\theta,t)}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}.$$

Since t is independent of x we know $\partial_x t = 0$, so we can drop the last term. For polar coordinates we know

$$r = \sqrt{x^2 + y^2} \;\;\&\;\; \theta = \arctan(y/x).$$

Therefore

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta,$$

and

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{y}{x}\right)' = \frac{1}{1+(y/x)^2}\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}.$$

Hence,

$$u_x = u_r\cos\theta - \frac{1}{r}u_\theta \sin\theta.$$

One repeats this to get u_{xx} and similarly for u_{yy}. After simplifying the result is

$$\alpha^2(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}) = u_t.$$

Now we try to find solutions. As before we suppose $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$. Plugging into the heat equation gives.

$$\alpha^2\left(R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta''T\right) = R\Theta T'.$$

Divide through by $\alpha^2 R\Theta T$ to get

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{\alpha^2}\frac{T'}{T}.$$

Since the right side depends only on t and the left side is independent of t, both sides are constant; call this constant $-\sigma$. We get

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\sigma = -\frac{\Theta''}{\Theta} \;\;\&\;\; T' + \alpha^2\sigma T = 0.$$

Thus, there is a constant β such that

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\sigma = -\frac{\Theta''}{\Theta} = \beta.$$

Thus we get

$$r^2R'' + rR' + (r^2\sigma - \beta)R = 0 \;\;\&\;\; \Theta'' + \beta\Theta = 0.$$

Now, we have there separate ODE's.

We shall make a simplifying assumption, that we will hopefully drop later. But for now assume $u(r,\theta,t)$ is independent of θ. This means $\Theta' = 0$ and hence so does Θ''. Thus $\beta\Theta(\theta) = 0$ So, unless the temperature is always zero, will require $\beta = 0$. The R equation becomes

$$r^2R'' + rR' + \sigma r^2 R = 0.$$

Let $\gamma^2 = \sigma$ and $p = \gamma r$. Also define $P(p) = R(p/\gamma) = R(r)$. Then

$$r\frac{dR(r)}{dr} = \frac{p}{\gamma}\frac{dP(p)}{dr} = \frac{p}{\gamma}\frac{dP}{dp}\frac{dp}{dr} = \frac{p}{\gamma}\frac{dP}{dp}\gamma = pP'(p).$$

Similarly, $r^2R(r) = p^2P(p)$. Thus our equation for R becomes

$$p^2P'' + pP' + p^2P = 0.$$

This is Bessel's Equation of Order Zero. It is solved in Section 5.7[3]. The general solution is

$$P(p) = C_1J_0(p) + C_2Y_0(p).$$

Switching back to R and r we get

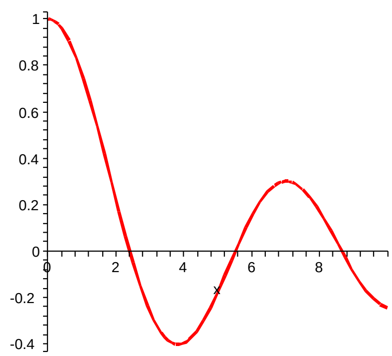
$$R(r) = C_1J_0(\gamma r) + C_2Y_0(\gamma r).$$

The functions J_0 and Y_0 are called Bessel functions of the first and second kind, respectively, both of order zero. They are determined by certain power series.

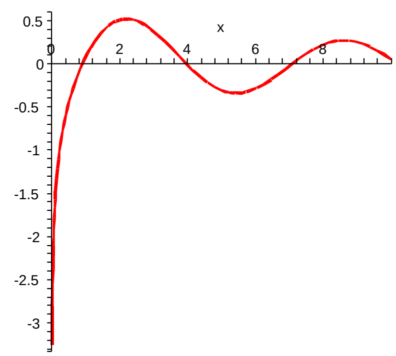
19.1 Bessel's Equations

Below are graphs of $J_0(x)$ and $Y_0(x)$.

J_0



Y_0



Next we consider the boundary conditions. First, $u(1, \theta, 0) = 0$ implies $R(1) = 0$. But this is a second "boundary" condition that is implicit: the temperature is bounded. In particular $|u(0, \theta, t)| < \infty$. Thus, $|R(0)| < \infty$. But, $Y_0(\gamma r)$ is unbounded as $r \to 0$. Hence $C_2 = 0$.

Now, $J_0(x)$ has infinitely many zeros. Enumerate them in increasing order by γ_1, γ_2, γ_3, etc., all positive. There are tables that list these or they can be generated by a computer. Here are the first twenty as generated by Maple.

```
> evalf(BesselJZeros(0,1..20));
```

2.404825558, 5.520078110, 8.653727913, 11.79153444, 14.93091771, 18.07106397, 21.21163663, 24.35247153, 27.49347913, 30.63460647, 33.77582021, 36.91709835, 40.05842576, 43.19979171, 46.34118837, 49.48260990, 52.62405184, 55.76551076, 58.90698393, 62.04846919

Then we let $R_n(r) = J_0(\gamma_n r)$. Each $R_n(r)$ satisfies the boundary conditions. Let

$$T_n(t) = e^{-\gamma_n^2\alpha^2 t} \quad \& \quad u_n(r,t) = X_n(r)T_n(t).$$

[3]Elementary Differential Equations & Boundary Value Problems 10th edition, by Boyce & DiPrima

Then each u_n satisfies the heat equation and the boundary conditions and so too would any linear combination. (We dropped θ since we are still assuming u is independent of θ.) It can be shown that this holds for

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\gamma_n r) e^{-\gamma_n^2 \alpha^2 t}$$

if the c_n's are such that the series convergences. Thus, if we can find c_n's such that

$$\sum_{n=1}^{\infty} c_n J_0(\gamma_n r) = u(r,0) = f(r)$$

we are done! It looks like the Fourier expansion with $J_0(\gamma_n r)$ in stead of $\sin(n\pi r)$ or $\cos(n\pi r)$. It is called a **Fourier-Bessel series**. Section 11.4 shows that this can be done and gives a formula for c_n.

$$c_n = \frac{\int_0^1 r f(r) J_0(\gamma_n r)\, dr}{\int_0^1 r J_0^2(\gamma_n r)\, dr}.$$

You should compare this to the formulas for the Fourier sine and cosine series. The chief difference is the integral in the denominator. For $\sin^2(n\pi x)$ and $\cos^2(n\pi x)$ the integral was independent of n and was just folded into the constants. For $J_0^2(\gamma_n r)$ this is not the case and that is why is has to be included as a normalizing term. The r in the integrals is a weighting factor that comes from using polar coordinates.

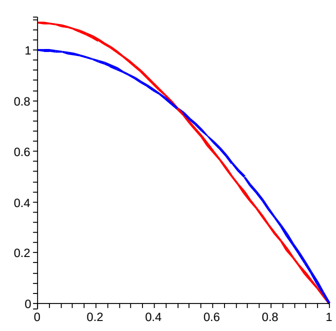
19.2 Example with Axial Symmetry

Suppose $f(r) = 1 - r^2$. We expand $f(r)$ as a series of Bessel functions. Then we plot a few partial sums to see how good the approximations are. We define `f(r)`, `g(n)` and `c(n)` with the Maple commands below; `g(n)` is γ_n and `c(n)` is c_n.

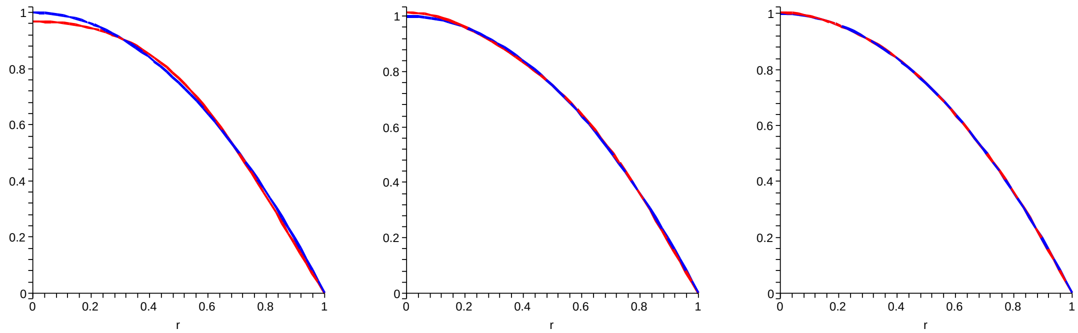
```
> f := r -> 1-r^2;
> g := n -> evalf(BesselJZeros(0,n)); # g(n) is n-th zero of Bessel function Jo.
> c := n -> (int(r*f(r)*BesselJ(0,g(n)*r) ,r=0..1))/(int(r*BesselJ(0,g(n)*r)^2 ,r=0..1));
```

Here is the plot for $N = 1$, that is we just polar the first term in the Fourier Bessel expansion.

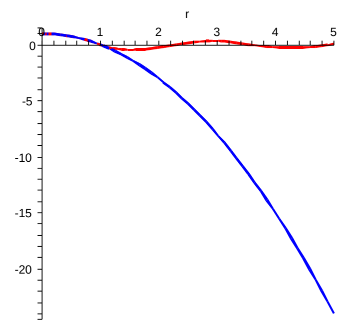
```
> N:=1:
> plot([sum(c(n)*BesselJ(0,g(n)*r),n=1..N),1-r^2 ] ,r=0..1,
thickness=2,color=[red,blue]);
```



We repeat for $N = 2, 3, \& 5$.



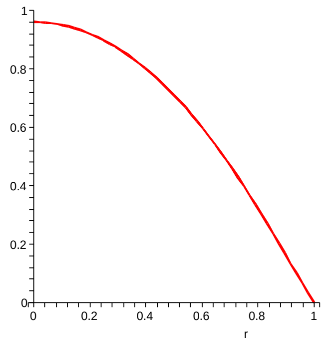
Looks good! But let's plot the graph out to $r = 5$; N is still 5.



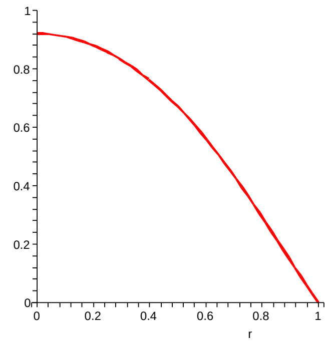
Yikes! Fortunately, we only care about $r \in [0, 1]$. Next we compute $u(r, t)$ and plot it for several values of t. I used $N = 10$ and then did plots for $t = 0.01, 0.02, 0.1, 0.2, 0.3,$ & 1.0. Only the command for $t = 0.01$ is shown.

```
> N:=10:t:=0.01:
> plot(sum(c(n)*BesselJ(0,g(n)*r)*exp(-g(n)^2*t),n=1..N) ,r=0..1,
thickness=2,color=red,view=0..1);
```

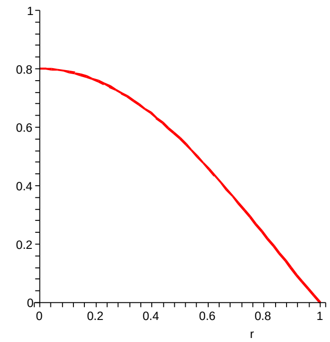
$t = 0.0$



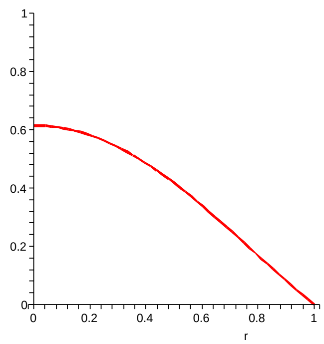
$t = 0.01$



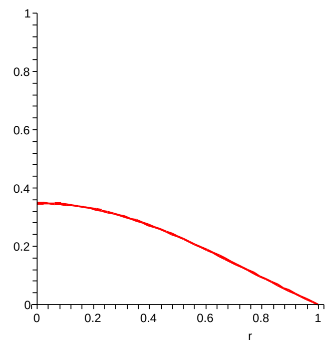
$t = 0.02$



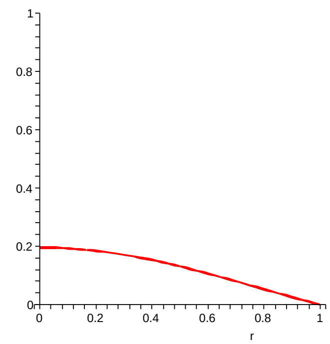
$t = 0.1$



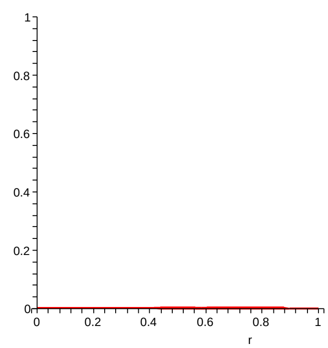
$t = 0.2$



$t = 0.3$



$t = 1.0$



19.3 Extra Credit!

Suppose $f(r) = r(1-r^2)$. The temperature at the origin starts at 0 and in the limit as $t \to \infty$ it will tend towards 0. What is the maximum temperature of the origin and when will it occur? Approximate it as best you can.

19.4 Example without Axial Symmetry

Suppose $\alpha = 1$ and $f(r,\theta) = r(1-r^2)\sin^2(3\theta)$. For the graphs here and below I used cylindrical coordinates, but Maple default plots r as a function of θ and z. Therefore, I had to define mine own cylindrical coordinates option where z is plotted as a function of r and θ.

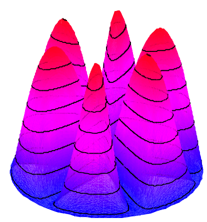
```
> addcoords(zcylindrical,[z,r,theta],[r*cos(theta),r*sin(theta),z],
[[1],[Pi],[0],[0..2,0..2*Pi,-1..1],[-4..4,-4..4,-4..4]])
```

I also created my own RGB color scheme just for fun.

```
B:= t -> piecewise(t<0.6, 1.0, -t/0.4 + 2.5 );
R:= t -> piecewise(t<0.4, t/0.4, 1.0);
```

Here then is a graph of $z = f(r, \theta)$.

```
> f := (r,theta) -> r*(1-r^2)*sin(3*theta)^2;
> plot3d(f(r,theta), r=0..1, theta=0..2*Pi,coords=zcylindrical,
view=0..0.5,color=,numpoints=10000);
```



We revisit the separated equations.

$$r^2R'' + rR' + (r^2\sigma - \beta)R = 0 \qquad \Theta'' + \beta\Theta = 0$$
$$T' + \sigma T = 0$$

The implied boundary condition on Θ is just $\Theta(0) = \Theta(2\pi)$. For $\beta < 0$ only the trivial solution is periodic. For $\beta = 0$, any constant solution will do. For $\beta > 0$ the general solution is

$$\Theta(\theta) = C_1 \cos \sqrt{\beta}\theta + C_2 \sin \sqrt{\beta}\theta.$$

Therefore, $\beta = n^2$ for $n > 0$. Thus, 1, $\sin n\theta$, and $\cos n\theta$, will solve the Θ equation and have an acceptable period as would any linear combination of these.

Now we go back to the R equation. Let $\gamma^2 = \sigma$, $\gamma > 0$, and consider

$$r^2R'' + rR' + (\gamma^2r^2 - n^2)R = 0.$$

The change of variable $p = \gamma r$ transforms this to

$$p^2P'' + pP' + (p^2 - n^2)P = 0.$$

This is Bessel's equation of order n. Its general solution takes the form

$$P(p) = C_1J_n(p) + C_2Y_n(p).$$

This is equivalent to

$$R(r) = C_1J_n(\gamma r) + C_2Y_n(\gamma r).$$

Here J_n is Bessel's function of the first kind of order n, and Y_n is Bessel's function of the second kind of order n, as you might have guessed. They are defined by certain power series (see *Introduction to Bessel Functions*, by Frank Bowman). Again, Y_n is unbounded at 0 and so $C_2 = 0$. Again J_n has infinitely many zeros. We denote them by γ_{nm}, $m = 1, 2, 3, \ldots$. They are positive and numbered in increasing order. Their numerical values can be found in tables or via a computer. The command in Maple for the m-th of J_n is `BesselJZeros(n,m)`.

For each $n = 0, 1, 2, 3...$ and $m = 1, 2, 3...$ let

$$R_{nm}(r) = J_n(\gamma_{nm} r).$$

Then $R_{nm}(1) = 0$ and $R_{nm}(r)$ solves $r^2 R'' + rR' + (\gamma_{nm}^2 r^2 - n^2)R = 0$. And of course we define

$$T_{nm} = e^{-\gamma_{nm}^2 t}.$$

Now, the products $R_{0m}(r)T_{0m}(r)$, $R_{nm}(r)T_{nm}(t)\cos(n\theta)$, and $R_{nm}(r)T_{nm}(t)\sin(n\theta)$, for $n, m > 0$, solve the heat equation and satisfy the boundary condition of being 0 when $r = 1$, are bounded, and have period 2π in θ. So too would any linear combination or convergent infinite sum. We suppose that,

$$u(r,\theta,t) = \frac{1}{2}\sum_{m=1}^{\infty} a_{0m}R_{nm}(r)T_{nm}(t) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}R_{n,m}(r)T_{nm}(t)\cos(n\pi) + b_{nm}R_{nm}(r)T_{nm}(t)\sin(n\pi)$$

converges. All that is left is to find values for the a_{nm} and b_{nm} such that

$$u(r,\theta,0) = f(r,\theta) = r(1-r^2)\sin^2(3\theta).$$

In our case we can use that

$$\sin^2(3\theta) = \frac{1}{2} - \frac{1}{2}\cos(6\theta)$$

Thus, only the a_{0m} and a_{6m} terms will be nonzero. We write

$$u(r,\theta,0) = r(1-r^2)\frac{1}{2} - r(1-r^2)\frac{1}{2}\cos(6\theta).$$

We need for

$$\sum_{m=1}^{\infty} a_{0m}R_{0m}(r) = \sum_{m=1}^{\infty} a_{0m}J_0(\gamma_{0m} r) = r(1-r^2)$$

and

$$\sum_{m=1}^{\infty} a_{6m}R_{6m}(r) = \sum_{m=1}^{\infty} a_{6m}J_6(\gamma_{6m} r) = r(1-r^2).$$

Each family of functions $\{J_n(\gamma_{nm} r)\}_{m=1}^{\infty}$ satisfies the orthogonality conditions needed to do series approximations. We get

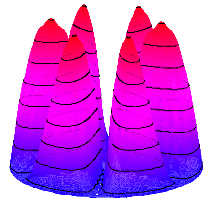
$$a_{0m} = \frac{\int_0^1 r^2(1-r^2)J_0(\gamma_{0m} r)\,dr}{\int_0^1 rJ_0^2(\gamma_{0m} r)\,dr}$$

and

$$a_{6m} = \frac{\int_0^1 r^2(1-r^2)J_6(\gamma_{6m} r)\,dr}{\int_0^1 rJ_6^2(\gamma_{6m} r)\,dr}.$$

Next we plot $u(r,\theta,0)$ and compare it to the graph for $f(r,\theta)$ above.

```
> fr := r -> r*(1-r^2);
> g := (n,m) -> evalf(BesselJZeros(n,m));
> a0 := m -> (int(r*fr(r)*BesselJ(0,g(0,m)*r),r=0..1))
/(int(r*BesselJ(0,g(0,m)*r)^2 ,r=0..1));
> a6 := m -> (int(r*fr(r)*BesselJ(6,g(6,m)*r),r=0..1))
/(int(r*BesselJ(6,g(6,m)*r)^2 ,r=0..1));
> U := (r,theta) -> 1/2*sum( a0(m)*BesselJ(0,g(0,m)*r) ,m=1..M) -
1/2*cos(6*theta)*sum(a6(m)*BesselJ(6,g(6,m)*r), m=1..M);
> M:=10:
> plot3d(U0(r,theta),r=0..1,theta=0..2*Pi,coords=zcylindrical,view=0..0.5,
color=[R(U0(r,theta)/0.4), 0, B(U0(r,theta)/0.4)],numpoints=10000);
```

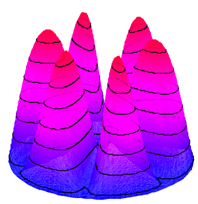


Now we put all the pieces together to get

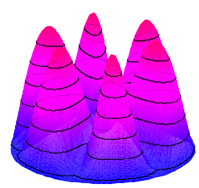
$$u(r,\theta,t) = \frac{1}{2}\sum_{m=1}^{\infty} a_{0m} J_0(\gamma_{nm} r) e^{-\gamma_{0m}^2 t} - \frac{1}{2}\cos(6\theta)\sum_{m=1}^{\infty} a_{6m} J_6(\gamma_{nm} r) e^{-\gamma_{6m}^2 t}.$$

We plot this for a few values of t.

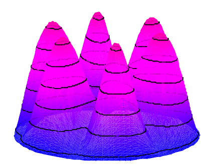
$t = 0.001$



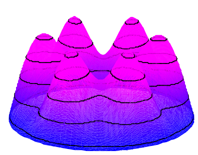
$t = 0.003$



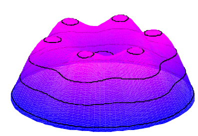
$t = 0.005$



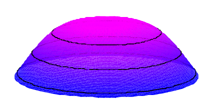
$t = 0.01$



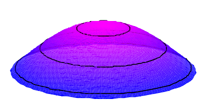
$t = 0.02$



$t = 0.05$



$t = 0.1$



19.5 General Case

In the last example we took advantage of the fact the given $f(r,\theta)$ was separable. This is not always the case. I found the online lecture notes "NOTE12: Fourier-Bessel Series and BVP in Cylindrical Coordinates" by Hamid Meziani, 2016, especially helpful at this point.
See `http://faculty.fiu.edu/~meziani/MAP4401.html`.

$$u(r,\theta,t) = \frac{1}{2}\sum_{m=1}^{\infty} a_{0m}R_{0m}(r)T_{0m}(t) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}R_{nm}(r)T_{nm}(t)\cos(n\theta) + b_{nm}R_{nm}(r)T_{nm}(t)\sin(n\theta)$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}J_n(\gamma_{nm}r)e^{-\gamma_{nm}^2 t}\cos(n\theta) + b_{nm}J_n(\gamma_{nm}r)e^{-\gamma_{nm}^2 t}\sin(n\theta).$$

At time $t=0$ we need

$$u(r,\theta,0) = \frac{1}{2}\sum_{m=1}^{\infty} a_{0m}J_0(\gamma_{0m}r) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}J_n(\gamma_{nm}r)\cos(n\theta) + b_{nm}J_n(\gamma_{nm}r)\sin(n\theta) = f(r,\theta).$$

For each value of $r \in [0,1]$ the function $f(r,\theta)$ is periodic with period 2π and so has a Fourier series with respect to θ. We compute the coefficients as normal, but regard them as functions of r.

$$f(r,\theta) = \frac{A_0(r)}{2} + \sum_{n=1}^{\infty} A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta)$$

where

$$A_n(r) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos(n\theta)\,d\theta,$$

and

$$B_n(r) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin(n\theta)\,d\theta,$$

for $n = 0,1,2,3,...$ (of course $B_0 = 0$ so we ignore it).

Now, each $A_n(r)$ and $B_n(r)$ has a Fourier-Bessel series in terms of $\{J_k(\gamma_{km}r)\}_{m=1}^{\infty}$ for any k, so we can choose $k=n$. We have

$$A_n(r) = \sum_{j=1}^{\infty} a_{nm}J_n(\gamma_{nm}r) \quad \& \quad B_n(r) = \sum_{j=1}^{\infty} b_{nm}J_n(\gamma_{nm}r),$$

where

$$a_{nm} = \frac{\int_0^1 rA_n(r)J_n(\gamma_{nm}r)\,dr}{\int_0^1 rJ_n^2(\gamma_{nm}r)\,dr} \quad \& \quad b_{nm} = \frac{\int_0^1 rB_n(r)J_n(\gamma_{nm}r)\,dr}{\int_0^1 rJ_n^2(\gamma_{nm}r)\,dr}.$$

Combining these with the integral formulas for $A_n(r)$ and $B_n(r)$ gives the following.

$$a_{nm} = \frac{1}{\pi}\frac{\int_0^1\int_0^{2\pi} rf(r,\theta)J_n(\gamma_{nm}r)\cos(n\theta)\,d\theta\,dr}{\int_0^1 rJ_n^2(\gamma_{nm}r)\,dr}$$

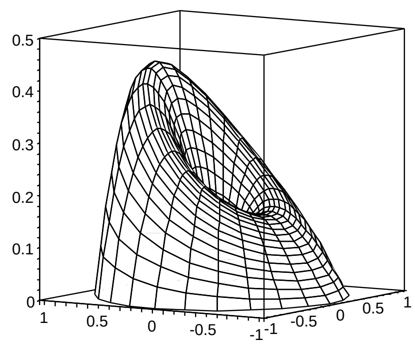
and

$$b_{nm} = \frac{1}{\pi}\frac{\int_0^1\int_0^{2\pi} rf(r,\theta)J_n(\gamma_{nm}r)\sin(n\theta)\,d\theta\,dr}{\int_0^1 rJ_n^2(\gamma_{nm}r)\,dr}.$$

19.6 Example: Not Axially Symmetric and Not Separable

Let $f(r,\theta) = r(1-r)e^{r\sin\theta}$. Its graph is below. I did not use color because the plots took too long.

```
>f := (r,theta) -> r*(1-r)*exp(r*sin(theta));
> plot3d(f(r,theta),r=0..1,theta=0..2*Pi,coords=zcylindrical,color=white,view=-0.001..0.5)
```

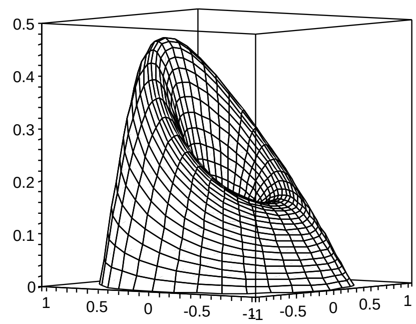


Here are the commands I used to construct the Fourier-Bessel series. First I defines the coefficients. Here `g(n,m)` is the m-th zero of J_n.

```
> g := (n,m) -> evalf(BesselJZeros(n,m));
> a := (n,m) -> evalf((1/Pi)*int(r*BesselJ(n,g(n,m)*r)*int( f(r,theta)*cos(n*theta),
theta=0..2*Pi), r=0..1)/ int(r*BesselJ(n,g(n,m)*r)^2,r=0..1));
> b := (n,m) -> evalf((1/Pi)*int(int( f(r,theta)*sin(n*theta)*r*BesselJ(n,g(n,m)*r),
theta=0..2*Pi),r=0..1)/ int(r*BesselJ(n,g(n,m)*r)^2,r=0..1));
```

Because these calculations take so long I stored the results in two data matrices, A and B. Since n starts at 0 we get $A[1,1] = a(0,1)$, and so on. The `sum` does not like entries that are from a matrix, so I had to use the `add` command instead. Next we plot $u(r,\theta,0)$ and compare with the plot of $f(r,\theta)$. I ended up only needing a few terms.

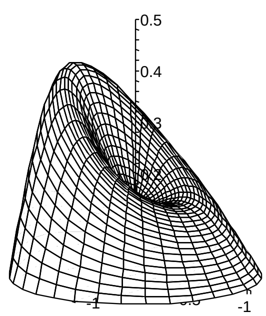
```
> N:=2:  M:=2:
> U0:= (r,theta) -> add(BesselJ(0,g(0,m)*r)*A[1,m] ,m=1..M)/2 +
add(add(BesselJ(n,g(n,m)*r)*(A[n+1,m]*cos(n*theta) + B[n+1,m]*sin(n*theta)),m=1..M ),
n=1..N );
> plot3d(U0(r,theta),r=0..1,theta=0..2*Pi,coords=zcylindrical,numpoints=1000,
color=white,view=-0.001..0.5);
```

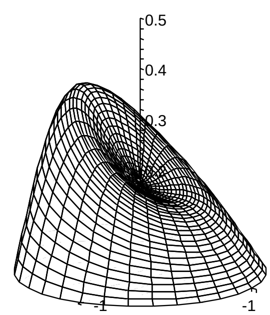


This looks pretty close. Next I defined our solution $u(r, \theta, t)$. Then I plotted in for several values of t.

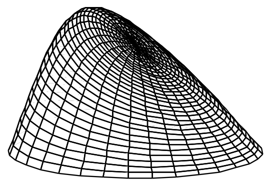
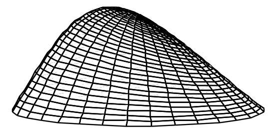
```
> U := (r,theta,t) -> add(BesselJ(0,g(0,m)*r)*A[1,m]*exp(-g(0,m)^2*t) ,m=1..M)/2 +
add(add(BesselJ(n,g(n,m)*r)*exp(-g(n,m)^2*t)*(A[n+1,m]*cos(n*theta) +
B[n+1,m]*sin(n*theta)),m=1..M ), n=1..N );
> plot3d(U(r,theta,0.01),r=0..1,theta=0..2*Pi,coords=zcylindrical,numpoints=1000,
color=white,view=-0.001..0.5);
```

$t = 0.01$ $t = 0.02$





$t = 0.05$ $t = 0.1$

Finally I did an animation. I'll put it on the web site. Here is the command that generates it.

```
> with(plots):
> animate(plot3d ,[ U(r,theta,a),r=0..1,theta=0..2*Pi,coords=zcylindrical,
numpoints=1000,color=white,view=-0.001..0.5 ] ,a=0.0..0.2,frames=200);
```

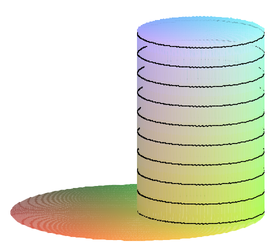
19.7 Example: Discontinuous Initial Condition

Suppose the initial temperature distribution is 1 inside a disk of radius 1/2 with center (0,1/2) and 0 elsewhere on the unit disk. Let

$$f(r,\theta) = \begin{cases} 1 & 0 \le \theta \le \pi \,\&\, r < \sin\theta \\ 0 & \text{otherwise} \end{cases}$$

See the graph below.

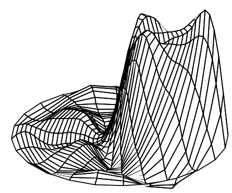
```
> plot3d(f,0..1,0..2*Pi,coords=zcylindrical,numpoints=5000) #The default coloring has
no physical significance.
```



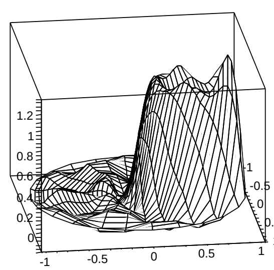
Next we compute the coefficients and plot $u(r,\theta,0)$.

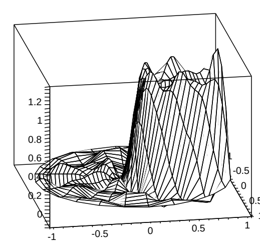
```
> g := (n,m) -> evalf(BesselJZeros(n,m));
> a := (n,m) -> evalf((1/Pi)*evalf(Int(evalf(Int( 1*r*BesselJ(n,g(n,m)*r)*cos(n*theta),
r=0..sin(theta))), theta=0..Pi))/int(r*BesselJ(n,g(n,m)*r)^2,r=0..1));
> b := (n,m) -> evalf((1/Pi)*evalf(Int(evalf(Int( 1*r*BesselJ(n,g(n,m)*r)*sin(n*theta),
r=0..sin(theta))), theta=0..Pi))/int(r*BesselJ(n,g(n,m)*r)^2,r=0..1));
```

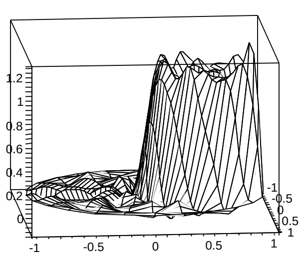
```
> U0:= (r,theta) -> add(BesselJ(0,g(0,m)*r)*(a(0,m)),m=1..M)/2 +
add(add(BesselJ(n,g(n,m)*r)*(a(n,m)*cos(n*theta)
+ b(n,m)*sin(n*theta)),m=1..M ), n=1..N );
> N:=4:  M:=4:
>plot3d(U0(r,theta),r=0..1,theta=0..2*Pi,coords=zcylindrical,color=white);
```

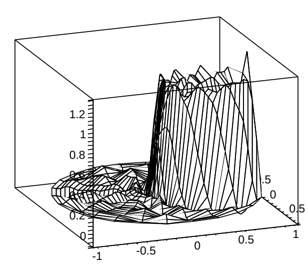


Unsurprisingly, convergence is much slow where the initial function is discontinuous. We plot several more partial sums: $N = M = 5$, $N = M = 6$, $N = 8, M = 6$, and $N = M = 9$. These plots take up a lot of computer time.









Next we define $u(r, \theta, t)$ and plot it for several values of t.

```
> U := (r,theta,t) -> add(BesselJ(0,g(0,m)*r)*exp(-g(0,m)^2*t)*(a(0,m)),m=1..M)/2 +
add(add(BesselJ(n,g(n,m)*r)*exp(-g(n,m)^2*t)*(a(n,m)*cos(n*theta) +
b(n,m)*sin(n*theta)),m=1..M ), n=1..N );
> N:=4:  M:=4:  t:=0.01;
> plot3d(U(r,theta,t),r=0..1,theta=0..2*Pi,coords=zcylindrical,color=white);
```

Also shown are the plots for $t = 0.02, 0.03$ and 0.06.