Part I: First Order Differential Equations

General Problem Let $F(x,y)$ be a given function of two variables. Find all functions $y(x)$ such that

$$\frac{dy}{dx} = F(x,y).$$

There are three special cases we will consider,

A. Separable: This means $F(x,y)$ can be written in the form $f(x)g(y)$. [See Section 2.2.]

B. Linear: This means $F(x,y)$ can be written in the form $f(x)y + g(x)$; so it is linear with respect to y. [See Section 2.1.]

C. Exact: This means $F(x,y)$ can be written in the form
$$F(x,y) = -\frac{M(x,y)}{N(x,y)} \quad \text{and} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$
[See Section 2.6.]

Sometimes when a first order diff. eq. is not given in any of these forms, it can be transformed into one that is by various algebraic tricks. However, there are many first order diff eq for which there is no ~~sp~~ special method and/or for which there is no closed form solution. For these we can use numerical methods [See Section 2.7].

I.A. Separable Equations

Ex 1 $y' = x^3$. Then $y = \int \frac{dy}{dx}\,dx = \int x^3 dx = \frac{1}{4}x^4 + C$,

which means the solution set is $\left\{ \frac{x^4}{4} + c \mid c \in \mathbb{R} \right\}$.

We say $y = \frac{x^4}{4} + C$ is the general solution.

Ex 2 $y' = \cos(2x) + 2$. Then $y = \int \cos(2x) + 2\, dx =$

$-\frac{1}{2}\sin(2x) + 2x + C$, is the general solution.

Rmk: In these two examples $y' = F(x,y)$ really only depended on x. Solving $y' = f(x)$ is just finding the anti-derivative or the indefinite integral.

Ex 3 $y' = \frac{x^2}{y}$. Rewrite as $y\frac{dy}{dx} = x^2$. Integrate

both sides with respect to x.

$\int y \frac{dy}{dx}\,dx = \int x^2 dx = \frac{1}{3}x^3 + C_1$, while

The LHS $= \int y\,dy = \frac{1}{2}y^2 + C_2$. Thus

$\frac{1}{2}y^2 + C_2 = \frac{1}{3}x^3 + C_1$. This gives

$y^2 = \frac{2}{3}x^3 + C$, where $C = 2(C_1 - C_2)$ is

still just an arbitrary constant.

Now $y = \pm\sqrt{\frac{2}{3}x^3 + C}$ is the general solution.

The solution set is,

$$\left\{ \sqrt{\tfrac{2}{3}x^3 + C} \;\middle|\; C \in \mathbb{R} \right\} \cup \left\{ -\sqrt{\tfrac{2}{3}x^3 + C} \;\middle|\; C \in \mathbb{R} \right\}.$$

<u>Ex 4</u> $y' = y^2 x^3$. Then $y^{-2} y' = x^3$. Integrate,

$$\int y^{-2} \frac{dy}{\cancel{dx}} \cancel{dx} = \int x^3 \, dx$$

$$-\frac{1}{y} = \frac{1}{4} x^4 + C$$

$$y = \frac{-1}{\frac{1}{4} x^4 + C}$$

$\frac{4}{C - x^4}$

which is equivalent to $y = -4/(x^4 + C)$.

<u>Ex 5</u> $y' = \frac{\sin x}{y+1}$. As a short cut write, $(y+1)dy = \sin(x)\,dx$

Then, $\int y+1 \, dy = \int \sin x \, dx$,

gives, $\frac{1}{2} y^2 + y = -\cos x + C$.

We can solve for y with the quadratic formula.

Rewrite this as $y^2+2y+2\cos x + C = 0$. (new $C = -2C$)
Using $a=1$, $b=2$, $c = 2\cos x + C$, in the quadratic formula gives

$$y = \left(-2 \pm \sqrt{4 - 8\cos x - 4C}\right)/2,$$

or

$$\boxed{y = -1 \pm \sqrt{C - 2\cos x}} \quad (\text{new } C = 1-C).$$

This is the general solution, meaning the solution set is

$$\left\{-1+\sqrt{C-2\cos x} \;\middle|\; C \geq 2\right\} \cup \left\{-1-\sqrt{C-2\cos x} \;\middle|\; C \geq 2\right\},$$

assuming we only want realvalued solutions.

Ex 6 [Important for many applications.]
Find the general solution to $y' = y$.

Solution $\frac{dy}{dx} = y$ can be rewritten as

$$\int \frac{1}{y}\,dy = \int 1\,dx.$$

Thus, $\ln|y| = x + C$,

$$y = e^{x+C} = e^{C} e^{x}$$

Thus, $y = \pm e^{C} e^{x}$ or $y = Ce^{x}$, (new $C = \pm e^{C}$).

Note: $y=0$ also works since $0' = 0$. Thus C can be any real number.

Initial value problems. Find the solution to

$$y' = F(x, y) \quad \text{and} \quad y(x_0) = y_0.$$

Ex 1° $y' = x^3$ and $y(1) = 4$. We know the general solution is $y = \frac{x^4}{4} + C$. Since we are given

$y(1) = 4$ and we have $y(1) = \frac{1}{4} + C$, it must be that $C = 3\frac{3}{4}$. Then

$$y = \frac{x^4 + 15}{4}$$

is called a particular solution.

Ex 5° $y' = \frac{\sin x}{y+1}$ and $y(0) = 17$.

The general solution is $y = -1 \pm \sqrt{C - 2\cos x}$.

Then $y(0) = -1 \pm \sqrt{C-2} = 17$

$$\pm\sqrt{C-2} = 18$$

so we must use the $+\sqrt{\ }$ not the $-\sqrt{\ }$.

$\sqrt{C-2} = 18 \qquad C - 2 = 324 \qquad C = 326.$

Thus $y = -1 + \sqrt{326 - 2\cos x}$ is the particular solution we require.

Here are two applied examples. (See Section 2.3)

Ex 7 Radio Active Decay. Suppose A decays into B with rate r. This means

$$\frac{dA}{dt} = -rA.$$

ⓐ Find the general solution for $A(t)$.

$$\int \frac{1}{A}\, dA = -\int r\, dt$$

$$\ln|A| = -rt + c$$

$$|A| = e^{-rt+c} = Ce^{-rt} \quad (\text{new } C \geq 0)$$

$$A = \pm Ce^{-rt}$$

or $$A = Ce^{-rt} \quad (C=0 \text{ will work}).$$

ⓑ If $A(0) = A_0$, find the particular solution.

$A_0 = A(0) = Ce^{-r0} = C$. Thus $C = A_0$, and

$$A(t) = A_0 e^{-rt}.$$

ⓒ Find the "half-life", that is solve $A(t) = A_0/2$.

$$\tfrac{1}{2}A_0 = A_0 e^{-rt}$$

$$\ln\tfrac{1}{2} = \ln e^{-rt} = -rt$$

$$-\ln 2 = -rt$$

$$t_{1/2} = t = \frac{\ln 2}{r}.$$

Ex 8 Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T_a - T)$$

where T is temperature at time t, T_a is the ambient temp. and $k \geq 0$ is a constant. We solve for $T(t)$.

$$\int \frac{1}{T_a - T}\, dT = \int k\, dt$$

$$-\ln|T_a - T| = kt + c$$

$$|T_a - T| = e^{-kt-c} = Ce^{-kt} \quad (\text{new } C)$$

$$T_a - T = \pm Ce^{-kt} = Ce^{-kt} \quad \left(\text{new } C,\ C = 0 \text{ works if } T = T_a\right)$$

Thus $T(t) = T_a - Ce^{-kt}$

Exercise Suppose a cup of coffee has $T(0) = 200°F$ and one minute later $T(1\text{ min}) = 190°F$. Assume $T_a = 70°$. When will the coffee be $150°F$, in minutes and seconds?

Answer: 3 min 38 seconds.

Example Using Maple 17

There are several computer algebra systems that can solve differential equations. Here is an example using Maple release 17. We will find the general solution to $y' = x^2 \cdot \cos^2(y)$. The command we will use is called dsolve. (Hit control m to input Maple commands. Then control t takes us to text mode.)

```
> dsolve(diff(y(x),x)=x^2*(cos(y(x))^2),y(x));
```

$$y(x) = \arctan\left(\frac{1}{3}x^3 + \_C1\right)$$

I'll explain the terminology. We write y(x) to indicate y is regarded as a function of x. Then diff(y(x),x) means the derivative of y(x) with respect to x. This is to be equal to our given function of x and y. The last y(x) means solve this diff eq for y(x). The answer is $acrtan\left(\frac{x^3}{3} + C\right)$. Maple generated constants start with an underscore symbol. It uses _C1 since higher order differential equations can generate several constants. The semicolon means execute the command.

Next will we find a particular solution for $y(\pi) = -1$.

```
> dsolve({diff(y(x),x)=x^2*(cos(y(x))^2),y(Pi)=-1},y(x));
```

$$y(x) = \arctan\left(-\frac{1}{3}\pi^3 + \frac{1}{3}x^3 - \tan(1)\right)$$

If you prefer floating point approximations use the evalf command.

```
> evalf(%);
```

$$y(x) = \arctan(-11.89283328 + 0.3333333333\, x^3)$$

The % symbol means use the most recent output.

Matlab

Many Computer Algebra Systems can solve differential equations. Here is an example using Matlab, version 6.5. First we find the general solution for ~~Exercise 6~~ with the dsolve command:

```
>> dsolve('Dy=cos(y)^2*x^2','x')

ans =

atan(1/3*x^3+C1)
```

Then we find the particular solution for $y(\pi)=-1$:

```
>> dsolve('Dy=cos(y)^2*x^2','y(pi)=-1','x')

ans =

atan(1/3*x^3-1/3*pi^3-tan(1))
```

If you prefer decimal approximations use the vpa command (very precise arithmetic):

```
>> vpa('atan(1/3*x^3-1/3*pi^3-tan(1))',5)

ans =

atan(.33333*x^3-11.893)
```

~~Computer Exercise Redo Exercises 1-5 and 7-9 using Matlab.~~

Existance and Uniqueness Issues (Section 2.4)

For a given initial value problem for a first order diff. eq. when do solutions exist and when can we be sure the solution is unique.

Ex 1 Consider $y' = \frac{1}{x}$ $y(0) = 3$. The general sol. is

$y = \ln|x| + c$. But then $y(0)$ is undefined.

1.

Ex 2 Consider $y' = y^{\frac{1}{3}}$, $y(0) = 0$. Then

$$\int y^{-\frac{1}{3}}\,dy = \int dx$$

$$\frac{3}{2} y^{\frac{2}{3}} = x + c$$

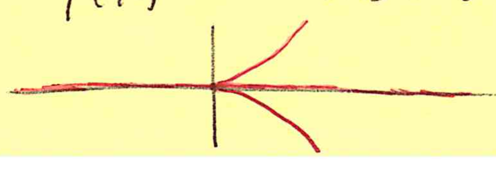
$$y = \left(\frac{2}{3}x + c\right)^{3/2} = \pm\sqrt{\left(\frac{2}{3}x + c\right)^3}.$$

$$y(0) = 0 \Rightarrow \pm\sqrt{\left(\frac{2}{3}\cdot 0 + c\right)^3} = 0 \Rightarrow c = 0.$$

But, this leave us with two solutions:

$$y = \left(\frac{2}{3}x\right)^{3/2} \qquad y = -\left(\frac{2}{3}x\right)^{3/2}.$$

Further more $y(x) = 0$ also works!



Theorem 2.4.2 (Picard's Thm)
Consider $y' = f(x,y)$ with $y(x_0) = y_0$.

If $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ both exist and are continuous in an open rectangle

$$\{(x,y) \mid a < x < b,\ c < y < d\}$$

that contains the point (x_0, y_0),

then there exists a unique function, $y(x)$, such that $y' = f(x,y)$ and $y(x_0) = y_0$.

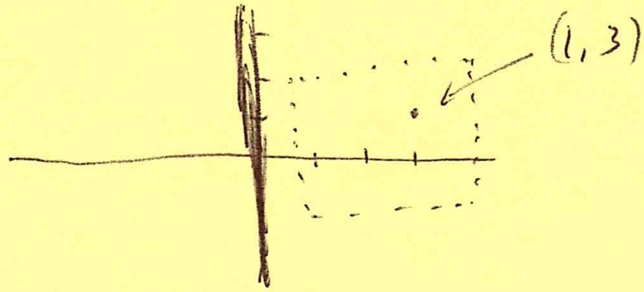
Furthermore, it is continuous on some open interval $(x_0 - h, x_0 + h) \subset (a,b)$, $(h > 0)$.

Note: This thm does not tell us how to find the solution or how big h can be.

Proof: The proof is done in Section 2.8 and you may read this if you wish.

Ex 1* Let's look back at Ex 1. Suppose the initial condition problem was: $y' = \frac{1}{x}$, $y(1) = 3$.

Then $f(x,y) = \frac{1}{x}$ is cont. for $x \neq 0$ and $\frac{\partial f}{\partial y} = 0$.



So, there should be a unique solution.

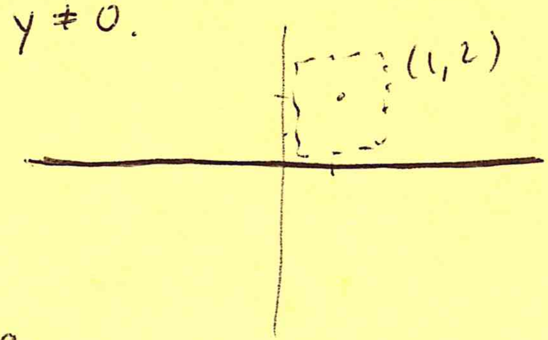
Sol $y = \ln|x| + C$. $y(1) = 3 \Rightarrow \ln 1 + C = 3$
$\Rightarrow C = 3$

$$y = \ln|x| + 3.$$

But it is only valid for $x > 0$.

Ex 2' Consider $y' = y^{\frac{1}{3}}$, $y(1) = 2$.

Now $f(x,y) = y^{\frac{1}{3}}$ and $\frac{\partial f}{\partial y} = \frac{1}{3} y^{-\frac{2}{3}}$ are cont. for $y \neq 0$.



Sol Next pg.

Sol $y = \pm\sqrt{\left(\frac{2}{3}x + C\right)^3}$. $y(1) = 2 \Rightarrow$

$2 = \pm\sqrt{\left(\frac{2}{3}\cdot 2 + C\right)^3}$. Thus we need + not −.

$2^2 = \left(\frac{4}{3} + C\right)^3$

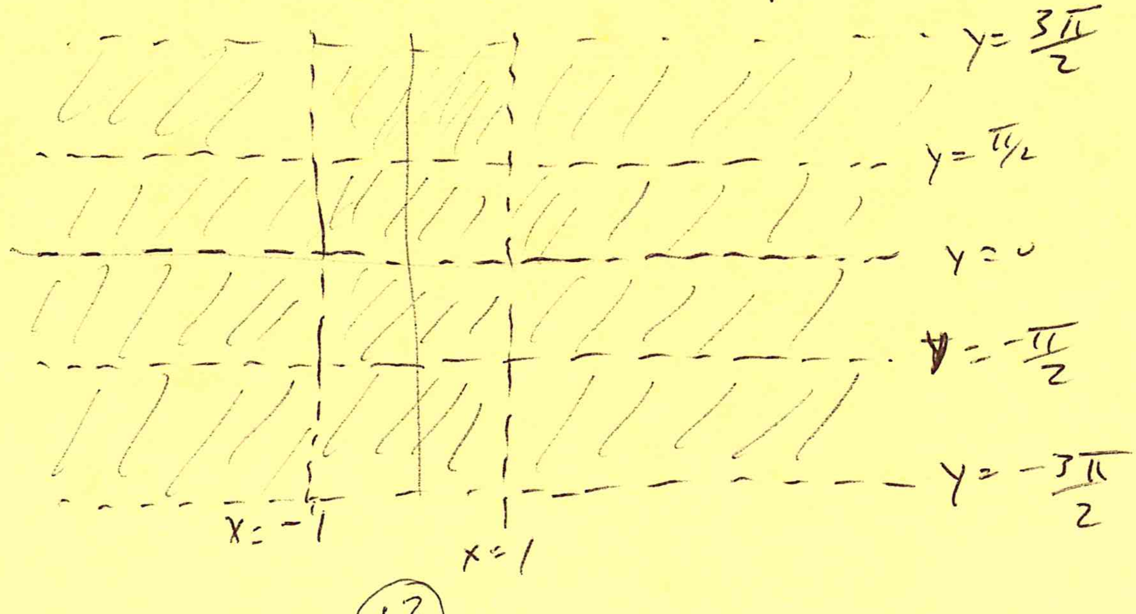
$2^{2/3} - \frac{4}{3} = C \approx 0.254067719$.

Ex Consider $y' = \dfrac{\tan(y)\cdot y^{2/5}}{x^2 - 1}$. Draw the set in $\mathbb{R}^2$ of init. cond.'s where a unique solution must exist.

Sol $x \neq \pm 1$ and $y \neq \frac{\pi}{2} + n\pi$, for all integers n.

Since $\dfrac{\partial}{\partial y} = \dfrac{\sec^2(y)\cdot y^{2/5}}{x^2 - 1} + \dfrac{\frac{2}{5}\tan(y)\, y^{-3/5}}{x^2 - 1}$

we need the restriction that $y \neq 0$.



Substitutions

Sometimes a diff eq. that is not separable can be transformed into one that is by an algebraic substitution. This is basically an extention of the u-substitution methods you learned in calculus. We illustrate this in the case of the so called Homogeneous Equations. (See pages 44-50 of the textbook.)*

Homogeneous Equations. First we do an example.

Ex 1 Consider $y' = e^{y/x} + \frac{y}{x}$. Let $v = y/x$. Then $y = xv$; Regarding v as a function of x we apply the prod rule:

$$y' = x'v + xv' = v + xv'.$$

Thus, we have $v + xv' = e^v + v$ or $xv' = e^v$. This is separable.

$$\int e^{-v}\,dv = \int \frac{1}{x}\,dx \implies$$

$$-e^{-v} = \ln|x| + C \implies$$

$$v = \ln(-\ln|x| + c) = \ln\left(\ln\left|\frac{1}{x}\right| + c\right).$$

Thus $y = xv = x\ln\left(\ln\left|\frac{1}{x}\right| + C\right)$ is the general solution.

* 10th ed.

In general, suppose $y' = F(x,y)$ and that $F(x,y)$ can be written as a function of y/x. That is $F(x,y) = f(y/x)$. Let $v = y/x$. Then

$$v + xv' = f(v), \text{ which is separable.}$$

$$\int \frac{dv}{f(v) - v} = \int \frac{1}{x}\,dx.$$

Then integrate, solve for v and use $y = xv$ to get $y(x)$, the general solution.

Def A function of two variables, $F(x,y)$, is called **homogeneous** if $F(tx,ty) = F(x,y)$. This is equivalent to saying $F(x,y)$ only depends on the ratio of y and x. This gives a way to test $F(x,y)$ to see if it depends only on y/x.

Ex 2 Let $F(x,y) = \dfrac{x^2 + 3y^2}{2xy}$. Then

$$F(tx,ty) = \frac{t^2x^2 + 3t^2y^2}{2t^2xy} = \frac{x^2 + 3y^2}{2xy} = F(x,y).$$

Thus $F(x,y)$ is homogeneous. It may take some effort to put $F(x,y)$ into the form $f(y/x)$. In this example,

$$F(x,y) = \frac{1}{2}\frac{x}{y} + \frac{3}{2}\frac{y}{x} = \frac{1}{2}\frac{1}{v} + \frac{3}{2}v = \frac{1+3v}{2}.$$

Ex 3 Let $F(x,y) = \sin(xy)$. It is not homogeneous since

$$\sin(tx \cdot ty) \neq \sin(xy).$$

$\left[\text{Let } x = \frac{\pi}{2},\ y = 1,\ t = 2.\ \ \sin(xy) = 1,\ \ \sin(tx, ty) = 0.\right]$

Thus, this method ~~would~~ cannot be applied to $y' = \sin(xy)$.

Ex 4 Let's go back to example 2 and find the general solution.

Consider $y' = \frac{x^2 + 3y^2}{2xy} = \frac{1}{2}\left(\frac{x}{y} + 3\frac{y}{x}\right) = \frac{1}{2}\left(\frac{1}{v} + 3v\right)$

where $v = y/x$. Since $y = xv \Rightarrow y' = v + xv'$, we have

$$v + xv' = \frac{1}{2}\left(\frac{1}{v} + 3v\right).$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1}{2}\left(\frac{1}{v} + 3v\right) - v = \frac{1}{2v} + \frac{v}{2} = \frac{1}{2}\left(\frac{1}{v} + v\right).$$

$$\Rightarrow \quad \int \frac{1}{\frac{1}{v} + v}\, dv = \int \frac{1}{2x}\, dx = \frac{1}{2}\ln|x| + C$$

$$\int \frac{v}{1+v^2}\, dv = \frac{1}{2}\ln(1+v^2) + C$$

Let $u = 1 + v^2$. $du = 2v\,dv$

Thus $\frac{1}{2}\ln(1+v^2) = \frac{1}{2}\ln|x| + c$

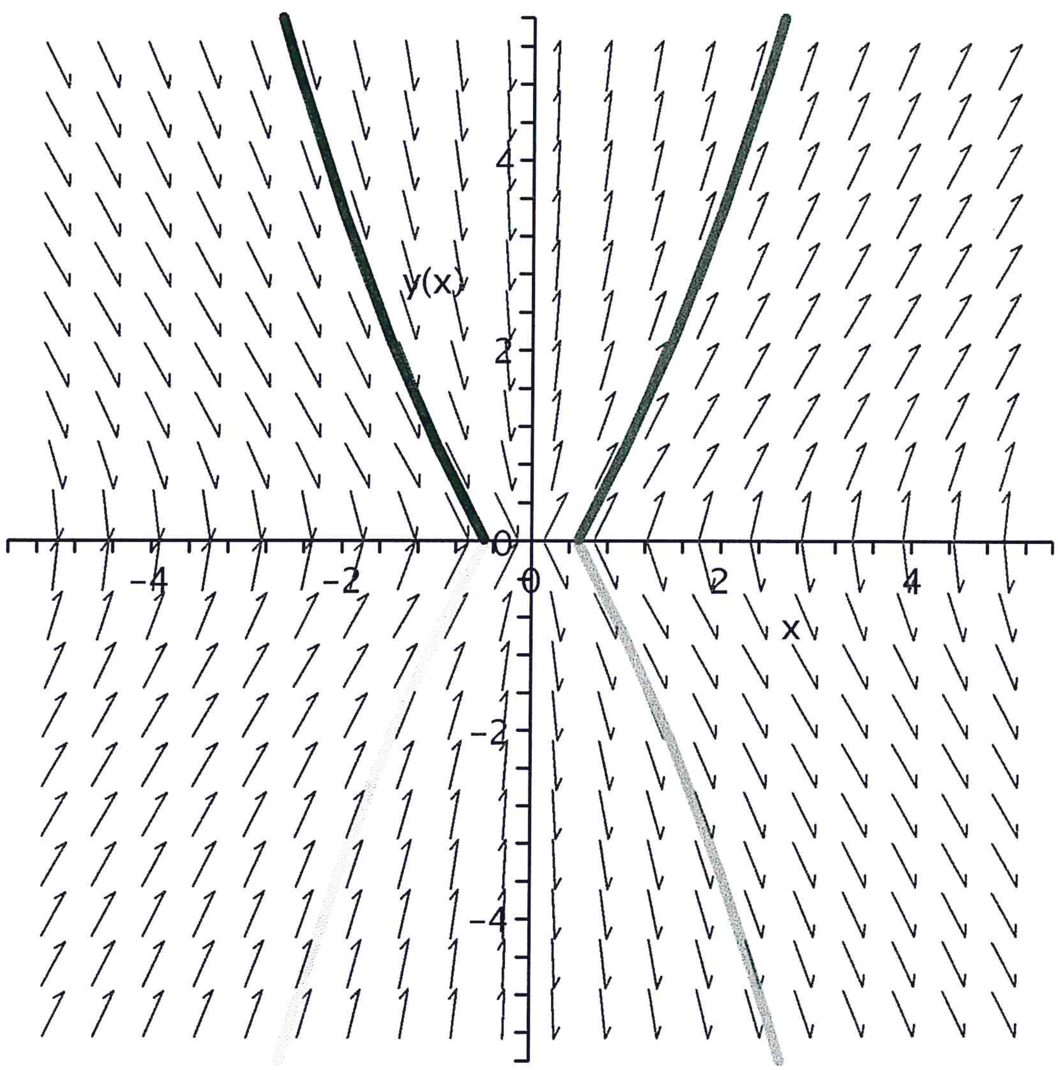
$$1+v^2 = e^{\ln|x|+c} = e^c|x| = Cx$$

Note: $1+v^2 = Cx$ implies x is always positive or always neg.

$$v = \pm\sqrt{Cx-1} \Rightarrow y = \pm x\sqrt{Cx-1}.$$

Question: Our C came from $\pm e^c$ which is never zero. Does $C=0$ work? No, unless y can be imaginary!

```
> with(DEtools):
> DEplot(diff(y(x),x) =
  (x^2+3*y(x)^2)/(2*x*y(x)),y(x),x=-5..5,y=-5..5,[[y(1)=1],[y(1)=-1],[y(-1)
  1],[y(-1)=-1]], linecolor=[red,green,blue,yellow]);
Warning, plot may be incomplete, the following errors(s) were issued:
   division by zero
```



I. B.

2.1 First order linear diff. eqs. are of the form

$$y' = -p(x)\,y + g(x), \text{ or } y' + p(x)\,y = g(x).$$

That is $y' = f(x,y)$ is a linear function of y, but not necessarily of x.

The solution method involves a trick. We will illustrate it with an example first and then develop the general method.

Ex 1 $\quad y' + 2y = x$. (It is linear, it is not sep.)

Let $\mu(x) = e^{2x}$ and multiply through by $\mu(x)$.

$$e^{2x}y' + 2e^{2x}y = xe^{2x} \qquad (*)$$

Watch closely: $(e^{2x}y)' = e^{2x}y' + 2e^{2x}y$, by the product rule. Substituting into $(*)$ gives

$$(e^{2x}y)' = xe^{2x}.$$

Thus, $\quad e^{2x}y = \int xe^{2x}\,dx = \frac{1}{2}(x-1)e^{2x} + C.$

Hence $\boxed{y = \frac{x-1}{2} + Ce^{-2x}.}$

The $\mu(x)$ is called an integrating factor; it allowed us to integrate and then solve for y. But where did it come from?

To find μ consider what $\mu(x)$ must do. We need

$$(\mu y)' = \mu y' + \mu p y.$$

We know

$$(\mu y)' = \mu y' + \mu' y.$$

Thus we need $\mu' = \mu p$. This is a sep. eq.

$$\int \frac{1}{\mu}\, d\mu = \int p(x)\, dx$$

$$\ln|\mu| = \int p(x)\, dx$$

$$\mu = \pm e^{\int p(x)\, dx}.$$

Since either $+/-$ would work we will choose $+$. Thus,

$$\boxed{\mu = e^{\int p(x)\, dx}.}$$

Ex 2 $y' + 2xy = x$. (Note: this is sep. Do it that way too and compare.)

Let $\mu = e^{\int 2x\,dx} = e^{x^2+c}$. Any c will do, so let $c=0$. Thus $\mu = e^{x^2}$. Multiplying gives

$$e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}, \text{ or}$$

$$(e^{x^2}y)' = xe^{x^2}.$$

Thus $$e^{x^2}y = \int xe^{x^2}\,dx = \tfrac{1}{2}e^{x^2}+c$$

so $$\boxed{y = \tfrac{1}{2} + ce^{-x^2}.}$$

Ex 3 $y' + \frac{1}{x}y = 3$.

Then $\mu = \pm e^{\int \frac{1}{x}dx} = \pm e^{\ln|x|+c} = \pm e^c|x| = C|x|$

Because $\frac{1}{x}$ is undefined at $x=0$, we can choose $C=1$ for $x>0$ and $C=-1$ for $x<0$. So $\mu = x$.

[We are really just solving $\frac{\mu'}{\mu} = \frac{1}{x}$, so $\mu = x$ clearly works.]

So $\mu = x$ is multiplied to both sides of

$$y' + \frac{1}{x} y = 3 \quad \text{to give}$$

$$x y' + y = 3x.$$

$$(xy)' = 3x$$

$$xy = \int 3x\,dx = \frac{3}{2}x^2 + C$$

Thus $\boxed{y = \frac{3}{2}x + \frac{C}{x}.}$ $\square$

We can also add an initial value as in the next example.

<u>Ex</u> $xy' + 2y = -\sin x$ with $y(\frac{\pi}{2}) = 4$.

<u>Solution</u> $y' + \frac{2}{x} y = -\frac{1}{x} \sin x$, so it is now linear.

Let $\mu = e^{\int \frac{2}{x} dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2$.

Multiply through by $\mu = x^2$, to get

$$x^2 y' + 2xy = -x \sin x$$

$$(x^2 y)' = -x \sin x$$

$$x^2 y = -\int x \sin x \, dx = x\cos x - \sin x + C$$

Thus $\boxed{y = \dfrac{x\cos x - \sin x + C}{x^2}}$ is the general sol.

Now set $y(\frac{\pi}{2}) = 4$ and solve for C.

$$\frac{\frac{\pi}{2}\overbrace{\cos(\tfrac{\pi}{2})}^{0} - \overbrace{\sin(\tfrac{\pi}{2})}^{1} + c}{(\frac{\pi}{2})^2} = 4$$

$$C - 1 = 4\frac{\pi^2}{4} = \pi^2$$

$$C = 1 + \pi^2$$

Thus $\boxed{y = \dfrac{x\cos x - \sin x + 1 + \pi^2}{x^2}}$ is the particular solution.

What would hapen if $y(0) = 4$?

Ex $y' - 2xy = 4$ with $y(0) = 3$. Estimate $y(1)$.

Sol.

$$\mu = e^{-\int 2x\,dx} = e^{-x^2}$$

$$e^{-x^2} y' - 2x e^{-x^2} y = 4 e^{-x^2}$$

$$\left(e^{-x^2} y\right)' = 4 e^{-x^2}$$

$$e^{-x^2} y = \int 4 e^{-x^2}\,dx$$, but you cannot do this integral.

$$y(x) = \frac{\int_0^x 4 e^{-t^2}\,dt + C}{e^{-x^2}}$$

$$y(0) = \frac{\int_0^0 4 e^{-t^2}\,dt + C}{e^{-0^2}} = C$$

Thus $C = 3$.

$$y(1) = \frac{\int_0^1 4 e^{-t^2}\,dt + 3}{e^{-(1^2)}} = 4e \int_0^1 e^{-t^2}\,dt + 3.$$

```
> int(exp(-t^2),t=0..1);
```

$$\frac{1}{2}\,\mathrm{erf}(1)\sqrt{\pi}$$

```
> evalf(4*exp(1)*% + 3);
```

$$\boxed{11.12031388}$$

```
> evalf(4*exp(1)*int(exp(-t^2),t=0..1) + 3);
```

$$11.12031388$$

Existence and Uniqueness for First Order Linear Diff Eqs.

Consider $y' + p(x)y = g(x)$ $\quad y(x_0) = y_0 \quad (*)$

Thm 2.4.2 takes a particularly simple form:
Let $f(x,y) = -p(x)y + g(x)$. Then $\partial_y f = -p(x)$.

Thus, $p(x)$ and $g(x)$ are cont. iff $f(x,y)$ and $\partial_y f(x,y)$ are cont. Thus, $(*)$ has a unique solution whenever there is an open interval (a,b) containing x_0 on which $p(x)$ and $g(x)$ are cont.

Furthermore, it can be shown that the solution exists and is cont. on all of (a,b). See Thm 2.4.1 in Section 2.4.

<u>Ex</u> Consider $y' + \frac{y}{x-3} = \frac{1}{x}$. It will have a unique solution to $y(x_0) = y_0$ as long as $x_0 \neq 3$ or 0.

If $x_0 = 1$, the solution is valid on $(0,3)$.
If $x_0 = 5$, " " " " $(3,\infty)$.
If $x_0 = -\pi$ " " " " $(-\infty, 0)$.

Ex2 Consider $y' + \cot(x) y = x^3$, $y(1) = 77$. What is the domain of the solution?

Sol: $\cot(x)$ is undefined at $0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Notice $1 \in (0, \pi)$. Thus $y(x)$ is defined and cont. on $(0, \pi)$.

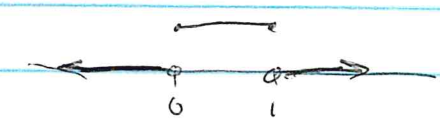
jump discontinuities only.

Linear Equations with discontinues Coeffiecent functions.

Ex We wish to find a continuous function $y(x)$ such that

$$y' + 2y = g(x) \quad , \quad y(0) = 0.$$

where $g(x) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \le 1 \\ 0 & 1 < t \end{cases}$



Solution: Step 1: $y' + 2y = 0. \quad (t \le 0)$

$$y' = -2y$$

$$y = Ce^{-2t}$$

$$y(0) = 0 \Rightarrow C = 0.$$

Step 2 $\quad y' + 2y = 1$

$$e^{2t}y' + 2e^{2t}y = e^{2t}$$

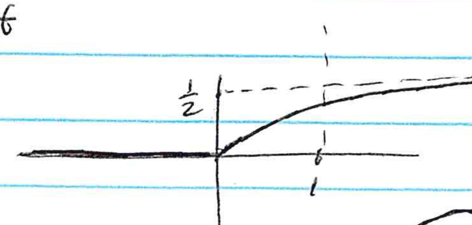
$$(e^{2t}y)' = e^{2t}$$

$$e^{2t}y = \frac{1}{2}e^{2t} + C$$

$$y = \frac{1}{2} + Ce^{-2t}$$

$$y(0) = 0 \Rightarrow C = -\frac{1}{2}.$$

$$y = \frac{1 - e^{-2t}}{2}$$



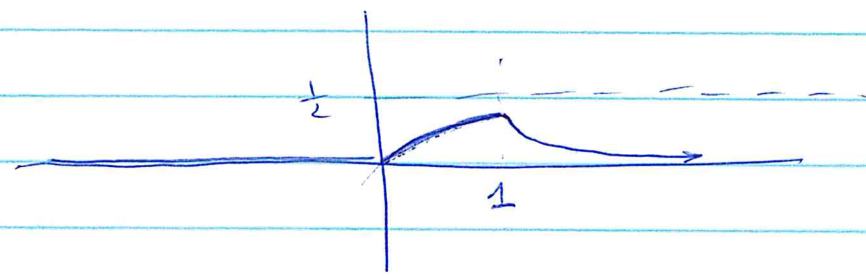
$(y'' = -2y' = -2e^{-2t} = -2,$
concave down)

Step 3 $\quad y' + 2y = 0 \quad y = Ce^{-2t}$

$$y_{old}(1) = \frac{1-e^{-2}}{2} = y_{new}(1)$$

$$y(1) = \frac{1-e^{-2}}{2} = Ce^{-2} \qquad \frac{8}{5} = \frac{4}{5}$$

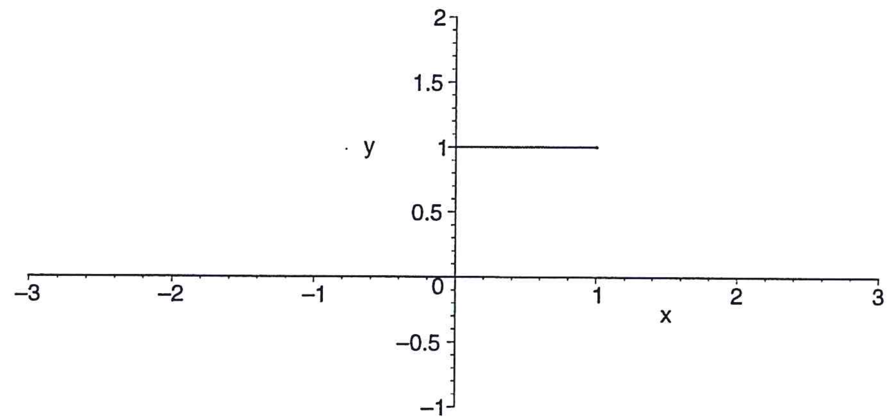
$$\boxed{\frac{e^{2}-1}{2} = C \approx 3.4}$$



radiation?

In class we solved y' +2y = g(x) where g(x) was a peicewise constant function; it is plotted below.

```
> plot(piecewise( x<0, 0, 0<=x and x<=1, 1, 1<x, 0),
  x=-3..3,y=-1..2,thickness=3, discont=true);
```



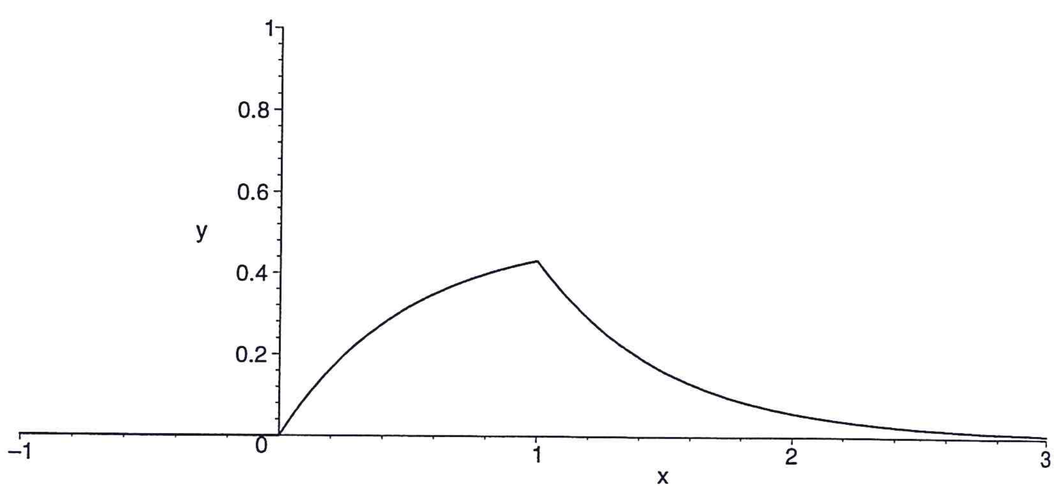
We can use dsolve in Maple to find the solution when y(0)=0 is the given initial condition.

```
> dsolve({diff(y(x),x)+2*y(x)=piecewise(x<0, 0, 0<=x and x<=1, 1,
  1<x, 0),y(0)=0});
```

$$y(x) = \left(\begin{cases} e^{(-2\,x)}\,\_C1 & x<0 \\ e^{(-2\,x)}\,\_C1 - \frac{1}{2}e^{(-2\,x)} + \frac{1}{2} & x<1 \\ e^{(-2\,x)}\,\_C1 - \frac{1}{2}e^{(-2\,x)} + \frac{1}{2}e^{(-2\,x+2)} & 1 \le x \end{cases} \right) \Bigg|_{\_C1=0}$$

Notice that the form of the answer is a bit awkward. We simplify it by inspection and graph it below.

```
> plot(piecewise(x<0,0,0<=x and x<=1, -1/2*exp(-2*x)+1/2, 1<x,
  -1/2*exp(-2*x)+1/2*exp(-2*x+2)), x=-1..3,y=0..1,thickness=3,
  color=blue);
```




```
>
>
```

See page 77 of the textbook.

Bernoulli Equations are equations of the form

$$y' + p(x)y = g(x)y^n \qquad (\#).$$

The substitution $v = y^{1-n}$ will transform (#) into the linear equation

$$v' + (1-n)p(x)v = (1-n)g(x). \qquad (\ast)$$

Ex 1 $y' + 2y = y^4$ (Note: this is separable, but hard to integrate)

[n=4]

Let $v = y^{-3}$. Then $y = v^{-1/3}$ and, by the chain rule

$$y' = -\tfrac{1}{3} v^{-4/3} v'. \quad \text{Thus we get}$$

$$-\tfrac{1}{3} v^{-4/3} v' + 2v^{-\frac{1}{3}} = v^{-4/3}$$

Multiply through by $-3v^{4/3}$ to get

$$v' - 6v = -3$$

This is linear in v. $v(x) = \frac{1}{2} + Ce^{6x}$. Thus

$$y(x) = \left(\tfrac{1}{2} + Ce^{6x}\right)^{-1/3}$$

is the general solution.

Ex 2 $t^2 y' + 2ty - y^3 = 0$. Assume $t > 0$.

Sol Convert to $y' + \frac{2y}{t} = \frac{y^3}{t^2}$. Thus, $n = 3$.

Let $v = y^{1-3} = y^{-2}$. Then $y = v^{-\frac{1}{2}}$. Thus $y' = -\frac{1}{2} v^{-\frac{3}{2}} v'$.

Thus,

$$-\frac{1}{2} v^{-\frac{3}{2}} v' + \frac{2}{t} v^{-\frac{1}{2}} = \frac{1}{t^2} v^{-\frac{3}{2}}.$$

Hence $v' - \frac{4}{t} v = -\frac{2}{t^2}$. This is linear.

Let $\cancel{\mu = t^{\frac{4}{t}}}$. $\mu = e^{-\int \frac{4}{t}\,dt} = e^{-4\ln|t|} = t^{-4}$

Thus $t^{-4} v' - \cancel{\tfrac{4}{t}}\, 4t^{-5} v = -2t^{-6}$

$$(t^{-4} v)' = -2t^{-6}$$

$$t^{-4} v = \int -2t^{-6}\,dt = \frac{2}{5} t^{-5} + C$$

$$v = \frac{2}{5} t^{-1} + Ct^{-4}.$$

Thus $y = \pm \sqrt{\frac{1}{\frac{2}{5}t^{-1} + Ct^{-4}}} = \boxed{\frac{\pm 1}{\sqrt{\frac{2}{5t} + \frac{C}{t^4}}}}$.

$$y' + p(x)\, y = g(x)\, y^n$$

Let $v = y^{1-n}$. Then $y = v^{\frac{1}{1-n}}$ and

$$y' = \frac{1}{1-n} v^{\frac{1}{1-n} - 1} v' = \frac{1}{1-n} v^{\frac{n}{1-n}} v'$$

$$\frac{1}{1-n} v^{\frac{n}{1-n}} v' + p(x)\, v^{\frac{1}{1-n}} = g(x)\, v^{\frac{n}{1-n}}$$

Multiply through by $1-n\, v^{\frac{-n}{1-n}}$ to get

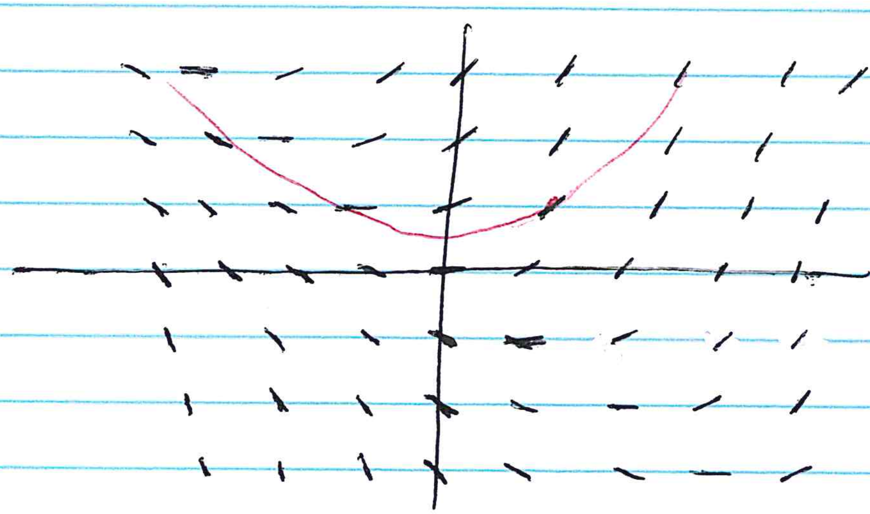
$$v' + (1-n)\, p(x)\, v^{\frac{1}{1-n} - \frac{n}{1-n}} = \cancel{g(x)}\, (1-n)\, g(x)$$

$$v' + (1-n)\, p(x) = (1-n)\, g(x)$$

Analysis of the solution

~~Direction~~ Slope Field.

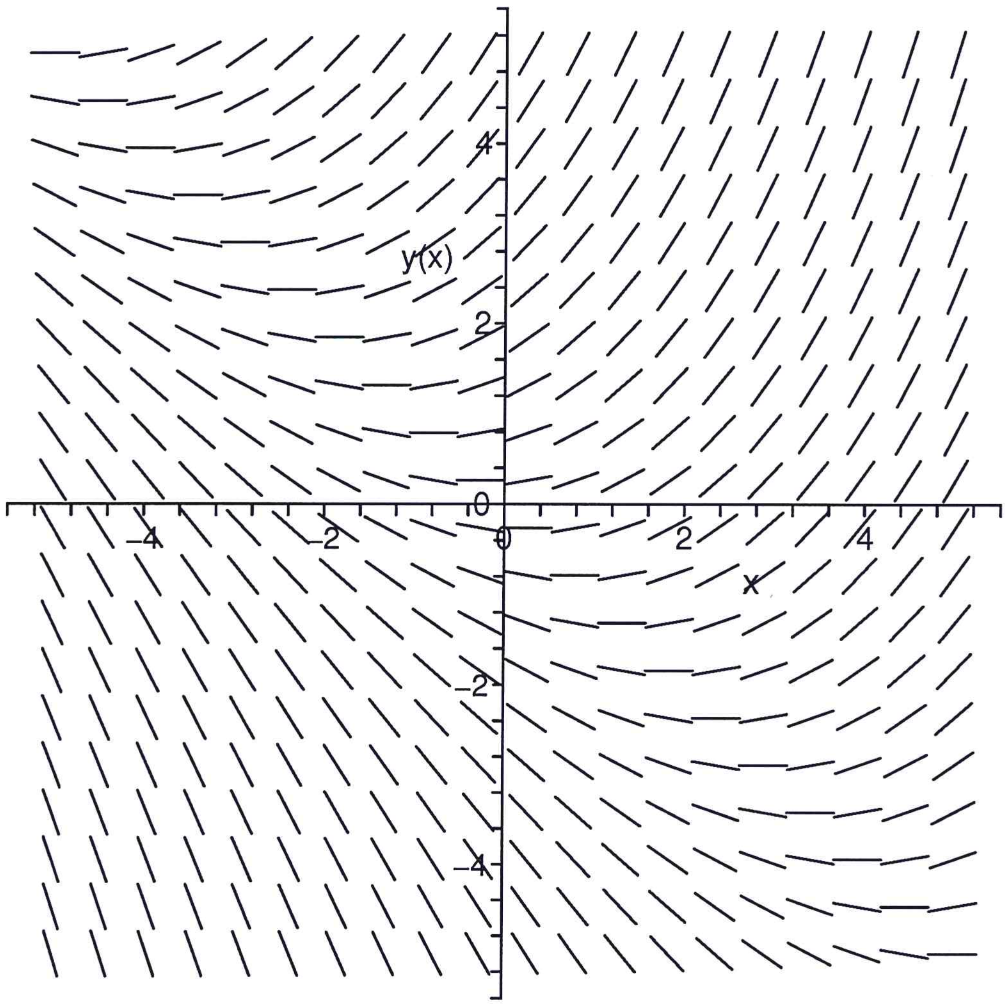
Ex Consider $y' = \frac{x+y}{3}$. This is easy to solve, but here we want to do something different. For each point (x_0, y_0) in the xy-plane we can compute y', the slope of a solution $f(x)$ with $f(x_0) = y_0$, even without knowing $f(x)$. At $(0,0)$ we get $y' = 0$, at $(2,1)$ we get $y' = \frac{2+1}{3} = 1$, etc. We shall place a small line segment at each grid location with a slope given by $y' = \frac{x+y}{3}$. See below:



Furtunately direction fields can be done by compute, or even on some calculators. First we use the "with" command to load a package of commands for working with differential equations. Then we employ the command "DEplot" to plot the direction field for dy/dx = (x+y)/3.

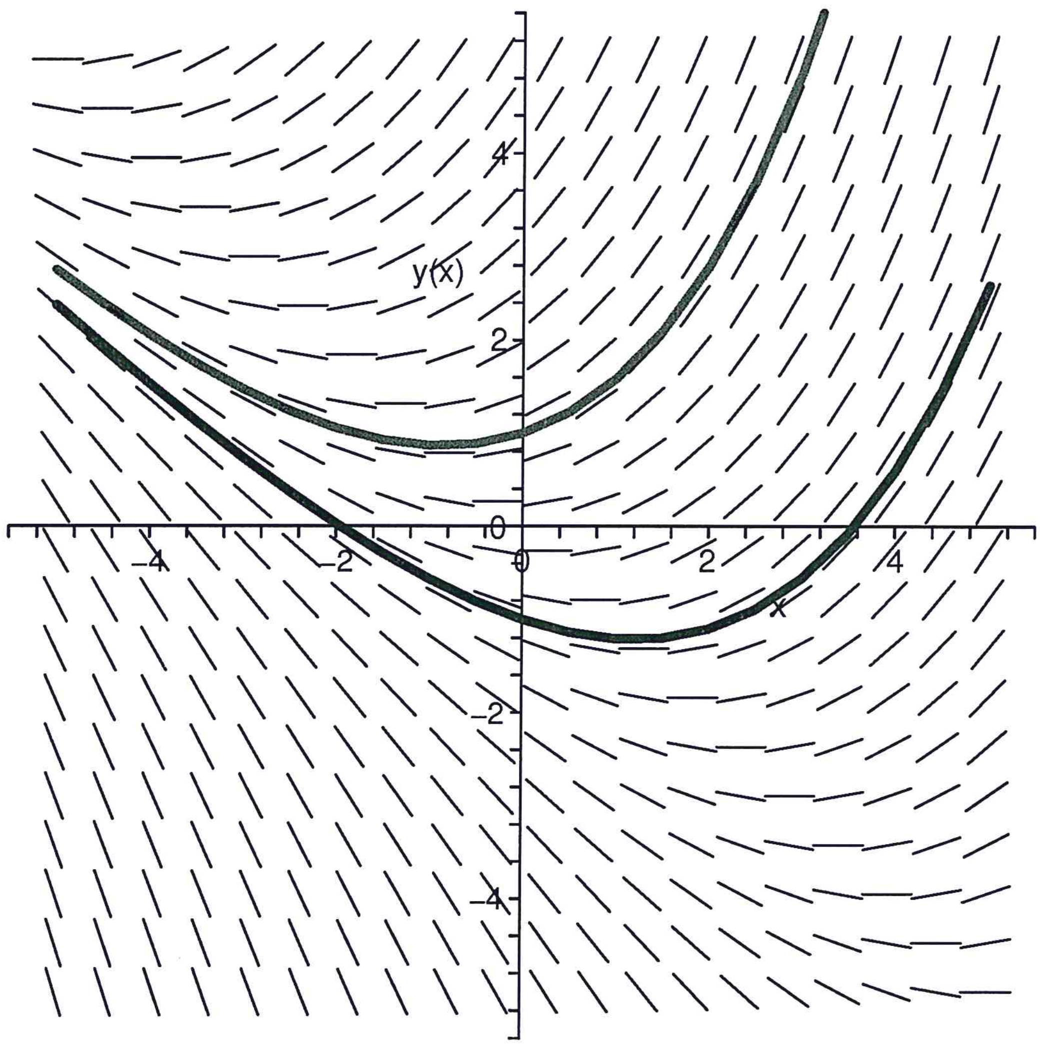
```
> with(DEtools):
```

```
> DEplot(diff(y(x),x) = (x+y(x))/3, y(x), x=-5..5,y=-5..5, arrows=line,
  color=black, thickness=3);
```



For our next trick we overlay solution curves for two different initial conditions, y(0) = 1 and y(0) =−1.

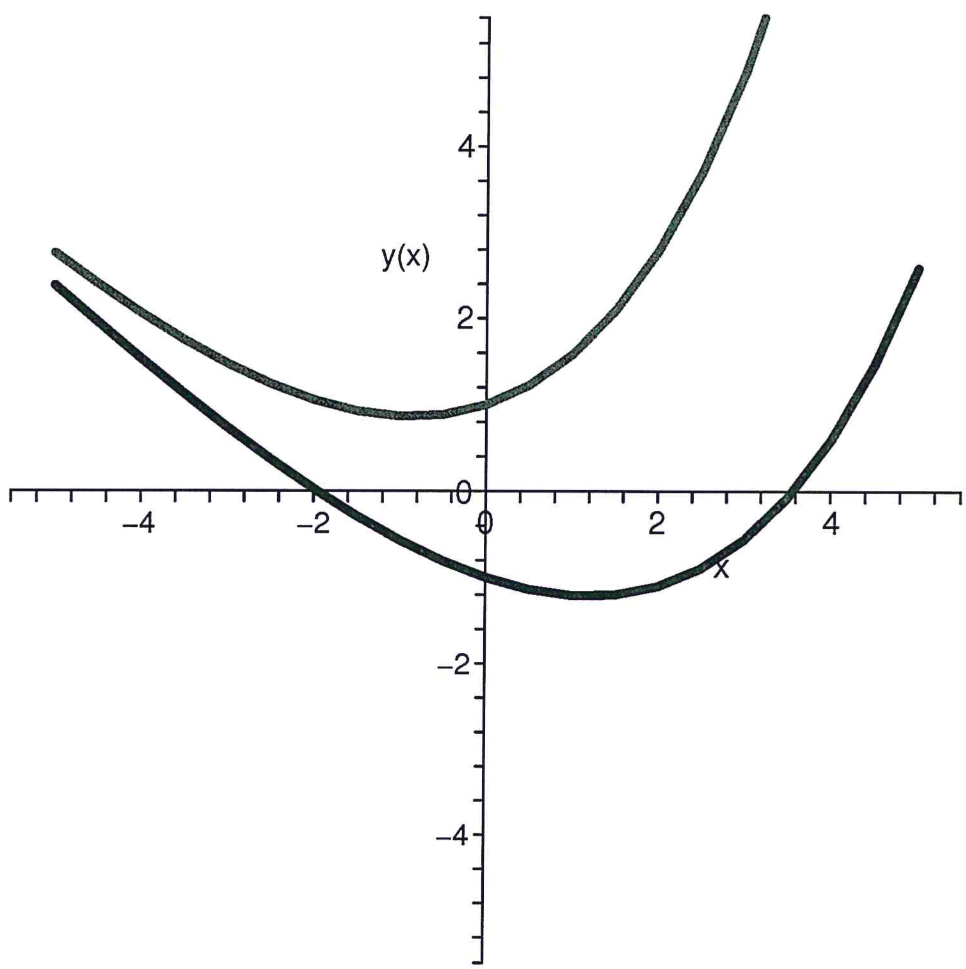
```
> DEplot(diff(y(x),x) = (x+y(x))/3, y(x), x=-5..5,y=-5..5,
  [[y(0)=1],[y(0)=-1]], arrows=line, color=black, thickness=3,
  linecolor=[red, blue]);
```



```
>
>
```

If you set arrows = none you get just the solution curves.

```
> DEplot(diff(y(x),x) = (x+y(x))/3, y(x), x=-5..5,y=-5..5,
  [[y(0)=1],[y(0)=-1]], arrows=none, color=black, thickness=3,
  linecolor=[red, blue]);
```




```
>
```

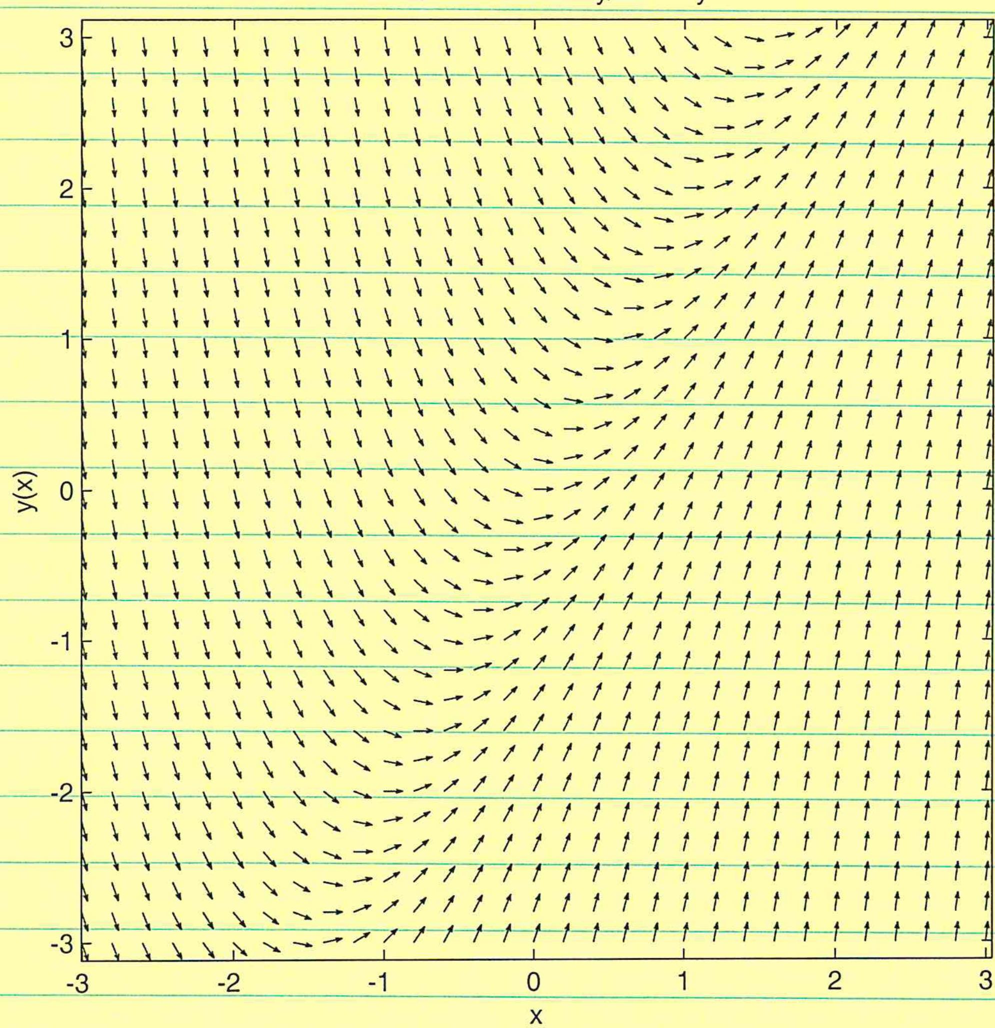
Here is how to make a direction field with Matlab. Let $y' = 2x - y$. See Ch 6, pages 75-79 ~~61-62~~ of the Matlab book for an explanation of the commands.

```
>> [x,y] = meshgrid(-3:0.2:3, -3:0.2:3);
>> slope = 2*x-y;
>> length = sqrt(1+slope.^2);
>> quiver(x,y,1./length,slope./length,0.5)
>> axis equal tight
>> xlabel 'x'
>> ylabel 'y(x)'
>> title 'Direction Field for dy/dx=2x-y'
>>
```

See output on next page.

~~Exercise 11: Redo Exercises 1-10 with Matlab.~~

Direction Field for dy/dx=2x-y



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