3.6 Variation of Parameters.

Suppose we want to solve $y'' + py' + qy = g$ (#) and that $\{y_1, y_2\}$ is a fundamental solution set for the homogeneous problem

$$y'' + py' + qy = 0. \qquad (*)$$

Thus $y_h = c_1 y_1 + c_2 y_2$ gives all solutions to $*$. Then a particular solution to the nonhomogeneous problem (#) is given by

$$y_p = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

where $$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W(y_1, y_2)}\, dt$$ and

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W(y_1, y_2)}\, dt.$$

Then the general solution is

$$y = c_1 y_1 + c_2 y_2 + y_p.$$

The formulas for u_1 and u_2 will be ~~derived~~ derived later. First we do a couple of examples. They get messy.

Ex Solve $y'' + y = \tan(t)$. $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

Sol We know $y_h = C_1 \sin(t) + C_2 \cos(t)$ is the general solution for $y'' + y = 0$. Let $y_1 = \sin t$ and $y_2 = \cos t$. Recall $W(y_1, y_2) = -1$. Thus

$$u_1 = -\int \frac{\cos t \tan t}{-1}\, dt = \int \sin t \, dt = -\cos t.$$

$$u_2 = \int \frac{\sin t \tan t}{-1}\, dt = -\int \frac{\sin^2 t}{\cos t}\, dt =$$

$$\int \frac{\cos^2 t - 1}{\cos t}\, dt = \int \cos t - \sec t \, dt$$

$$= \sin t - \ln(\sec t + \tan t)^{*}$$

Thus $y_p = u_1 y_1 + u_2 y_2 = -\cos t \sin t + \sin t \cos t - \cos t \ln(\sec t + \tan t)$

$$= -\cos t \ln(\sec t + \tan t).$$

Now, we check this to be sure we did not make a mistake.

* Look up the derivation of this integral formula.

$$y_p = -\cos t \ln(\sec t + \tan t)$$

$$y_p' = \sin t \ln(\sec t + \tan t) - \cos t \, \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t}$$

$$= \sin t \ln(\sec t + \tan t) - 1$$

$$y_p'' = \cos t \ln(\sec t + \tan t) + \sin t \, \frac{\sec t \tan t + \sec^2 t}{\sec t + \tan t}$$

$$= \cos t \ln(\sec t + \tan t) + \tan t$$

$$y_p'' + y_p = \cos t \ln(\sec t + \tan t) + \tan t - \cos t \ln(\sec t + \tan t)$$

$$= \tan t.$$ Yea!!

Now, the general solution is

$$\boxed{y = C_1 \sin t + C_2 \cos t - \cos t \ln(\sec t + \tan t) \quad \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}.}$$

Ex 2 $t^2 y'' - 2y = 3t^2 - 1$, $t > 0$, $y(1) = 0$, $y'(1) = 2$.

Sol First we try $y = t^n$ for the homogeneous problem

$$t^2 y'' - 2y = 0 \qquad (*)$$

$y'' = n(n-1)t^{n-2}$. So $(*)$ becomes

$$n(n-1)t^n - 2t^n = 0$$

$$(n^2 - n - 2)t^n = 0$$

$$(n-2)(n+1) = 0. \quad \text{Let } n = 2, -1.$$

Then $y_h = C_1 t^2 + C_2 t^{-1}$ is the general solution to $(*)$.

Put the original problem into standard form

$$y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2}.$$

We see that the method of undetermine coeff's does not apply.

Thus, this is a job for the variation of parameters!

Let $y_1 = t^2$ and $y_2 = t^{-1}$. $W(y_1, y_2) = -3$, as you can check!

$$u_1 = -\int \frac{\frac{1}{t}\left(3 - \frac{1}{t^2}\right)}{-3}\,dt = \frac{1}{3}\int 3t^{-1} - t^{-3}\,dt = \ln t + \frac{1}{6}t^{-2}$$

$$u_2 = \int \frac{t^2\left(3 - \frac{1}{t^2}\right)}{-3}\,dt = \int -t^2 + \frac{1}{3}\,dt = -\frac{1}{3}t^3 + \frac{1}{3}t$$

$$y_p = u_1y_1 + u_2y_2 = t^2\ln t + \frac{1}{6} \underbrace{- \frac{1}{3}t^2}_{} + \frac{1}{3}$$

Ignore this. Why?

Let $y_p = t^2\ln t + \frac{1}{2}$

General solution is $y = C_1t^2 + C_2t^{-1} + t^2\ln t + \frac{1}{2}$.

<u>Find C_1 & C_2.</u>

$$y(1) = C_1 + C_2 + \frac{1}{2} = 0$$
$$y'(t) = 2C_1t - C_2t^{-2} + 2t\ln t + \frac{t^2}{t} + 0$$
$$y'(1) = 2C_1 - C_2 = \cancel{2}\ -1$$

$$C_1 + C_2 = -\frac{1}{2}$$
$$2C_1 - C_2 = \cancel{2}\ -1$$

$$3C_1 = \frac{-3}{2} \Rightarrow C_1 = -\frac{1}{2} \Rightarrow C_2 = \cancel{1}\ 0$$

Final answer is

$$\boxed{y(t) = \frac{1}{2}t^2 - \cancel{t^{-1}} + \frac{1}{2} + t^2\ln t}$$

Ex Find general solution to $y'' + 6y' + 9y = \frac{1}{t}e^{-3t}$, $t > 0$.

Sol The characteristic poly is $r^2 + 6r + 9 = (r+3)^2$.
Thus the solution to the homogeneous problem

$$y'' + 6y' + 9y = 0$$

is $Y_h = C_1 e^{-3t} + C_2 t e^{-3t}$. $\left(\begin{array}{l} Y_1 = e^{-3t} \\ Y_2 = te^{-3t} \end{array} \right)$

$$W(e^{-3t}, te^{-3t}) = e^{-6t}$$ as you will surely check.

$$u_1 = -\int \frac{t e^{-3t} \cdot \frac{1}{t} e^{-3t}}{e^{-6t}} dt = -\int 1 \, dt = -t$$

$$u_2 = \int \frac{e^{-3t} \frac{1}{t} e^{-3t}}{e^{-6t}} dt = \int \frac{1}{t} dt = \ln t$$

Let $Y_p = \underbrace{-te^{-3t}}_{\text{Ignore this. Why?}} + t \ln t \, e^{-3t}$

Use $Y_p = t \ln t \, e^{-3t}$.

I will plug this into the original equation to test it.

$$y_p = t \ln t \, e^{-3t}$$

$$y_p' = \ln t \, e^{-3t} + e^{-3t} - 3t \ln t \, e^{-3t}$$

$$y_p'' = \frac{1}{t} e^{-3t} - 3 \ln t \, e^{-3t} - 3e^{-3t} - 3 \ln t \, e^{-3t} - 3e^{-3t} + 9t \ln t \, e^{-3t}$$

Then $y_p'' + 6y_p' + 9y_p =$

$$\left[9t \ln t + 6 \ln t + 6 - 18t \ln t + \frac{1}{t} - 3 \ln t - 3 - 3 \ln t - 3 + 9t \ln t \right] e^{-3t}$$

$$= \frac{1}{t} e^{-3t}. \quad \checkmark$$

[Note: It took me three tries to get this one!]

Thus, the general solution is

$$\boxed{y(t) = c_1 e^{-3t} + c_2 t e^{-3t} + t \ln t \, e^{-3t}.}$$

Derivation of the Formulas for u_1 and u_2.

Consider $y'' + p(t)y' + q(t)y = g(t)$. Suppose

$y_1(t)$ and $y_2(t)$ are linearly independent solutions of the corresponding homogeneous problem.
Let

$$y_p = u_1(t)\,y_1(t) + u_2(t)\,y_2(t).$$

Then $y_p' = u'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$.

We make a simplifying assumption that will turn out to be vadil. We assume

$$u_1'y_1 + u_2'y_2 = 0.$$

Now

$$y_p' = u_1y_1' + u_2y_2'$$

Then $$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Plug these into the original problem.

Oops! No primes!

$$u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + pu_1y_1' + pu_2y_2' + qu_1^{\cancel{\prime}}y_1 + qu_2^{\cancel{\prime}}y_2 = g$$

Regroup these terms to get ....

$$u_1(\underbrace{y_1''+py_1'+qy_1}_{=0}) + u_2(\underbrace{y_2''+py'+qy}_{=0}) + u_1'y_1' + u_2'y_2' = g$$

So, now we have

$$u_1'y_1' + u_2'y_2' = g \qquad (1)$$

We combine this with our simplifying assumption:

$$u_1'y_1 + u_2'y_2 = 0 \qquad (2)$$

Since y_1, y_2, y_1', y_2' are known, we have two equations and two unknowns: u_1', u_2'.

To solve for u_1' multiply (1) by y_2 and (2) by y_2', then substract.

$$u_1'y_1'y_2 + \boxed{u_2'y_2'y_2} = gy_2$$

$$u_1'y_1y_2' + \boxed{u_2'y_2y'} = 0$$

$$u_1'(y_1'y_2 - y_1y_2') = gy_2$$

$$u_1' = \frac{gy_2}{y_1'y_2 - y_1y_2'} = -\frac{y_2 g}{W(y_1,y_2)}.$$

You show $u_2' = \dfrac{y_1 g}{W(y_1,y_2)}$.