The Theory of Second Order Linear Differential Equations [1]

Michael C. Sullivan
Math Department
Southern Illinois University

These notes are intended as a supplement to section 3.2 of the textbook *Elementary Differential Equation and Boundary Value Problems*, by Boyce and DiPrima, 10th edition.

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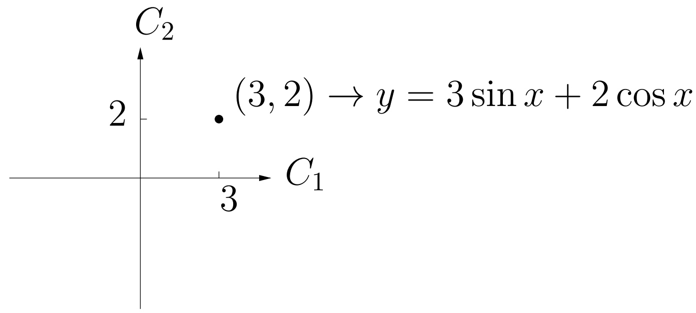
1. A motivating example

Consider the equation $y'' + y = 0$. We can see by inspection that $y_1 = \sin x$ and $y_2 = \cos x$ are solutions. It is easily checked that $y = C_1 \sin x + C_2 \cos x$ is a solution for all choices of the constants C_1 and C_2. Furthermore, **as the reader should check**, for all initial conditions of the from $y(x_0) = \alpha$ and $y'(x_0) = \beta$, we can find C_1 and C_2 such that $y = C_1 \sin x + C_2 \cos x$ satisfies these conditions. We will see later that we do indeed have all possible solutions to $y'' + y = 0$.

Now consider the figure below. It is a plane, but the axes are labeled C_1 and C_2. To each point in this plane we associate the function $y = C_1 \sin x + C_2 \cos x$. In this way the solution set of $y'' + y = 0$ becomes a *vector space*. We shall not give a formal definition of a vector space. But we note that, geometrically, adding pairs of solutions to our differential equation is just like adding vectors in this plane. Thus, the tools of linear algebra will come into play.


C_2
2
$(3, 2) \to y = 3\sin x + 2\cos x$
C_1
3

2. Review of 2×2 systems of equations

The *determinant* of a 2×2 matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Determinants are often denoted by replacing the brackets with straight lines:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Now, consider a 2×2 system of linear equations,

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \qquad (1)$$

and assume they represent two lines in the xy-plane. This can be rewritten in matrix notation as,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

We shall separate our discussion into two cases.

CASE I: When $e = f = 0$, we say the system of equations (1) is *homogeneous*. The pair equations can be thought of as representing two lines passing through the origin. Hence, if they have the same slope there are infinitely many points where they intersect. But if they have different slopes the only intersection point is the origin. It is easy to show that the slopes are the same if and only if the determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is zero. (Show this, and note that this result does not depend on e and f being zero.) Thus, we can state the following:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \iff \left\{ \begin{array}{l} \text{There are infinitely many} \\ \text{solutions to equation (1).} \end{array} \right\}$$

and

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \iff \left\{ \begin{array}{l} \text{There is only one solution,} \\ x = y = 0, \text{ to equation (1).} \end{array} \right\}$$

Case II: If either e or f is not zero, then we say the system of equations (1) is *nonhomogeneous*. Now the lines given by the two equations are not both going through the origin. If the slopes are different there is still a unique point of intersection. But if the slopes are the same either the lines coincide as before giving infinitely many solutions, or they are disjoint parallel lines and have no points in common. Thus, we can write:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \iff \left\{ \begin{array}{l} \text{There are infinitely many solutions or} \\ \text{there are no solutions to equation (1).} \end{array} \right\}$$

and

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \iff \{\text{There is only one solution to (1).}\}$$

The diligent reader will check the conclusions we have reached for several examples and draw the relevant graphs.

3. Linear Independence and the Wronskian

We now define some tools from linear algebra that will be useful in our study of the solutions of second order linear differential equations.

Definition 1 (Linear Independence)**.** Two functions, $f(x)$ and $g(x)$, defined on the same open interval I are *linearly independent* if neither is a nonzero multiple of the other. This is the same as saying it is impossible to find real numbers, C_1 and C_2, not both zero, such that

$$C_1 f(x) + C_2 g(x) = 0$$

for all $x \in I$. If it is possible two find two such numbers, not both zero, then we say f and g are *linearly dependent.* Again, this means f is a multiple of g or vice versa.

Problem 1. Show that x and x^2 are linearly independent over the real line.

Problem 2. Show that $|x|$ and $2x$ are linearly dependent on $I = (2, 3)$ but are linearly independent on $I = (-1, 1)$.

Definition 2 (Wronskian)**.** Let $f(x)$ and $g(x)$ be differentiable functions. Then their *Wronskian* at x is defined to be the function

$$W(x) = W(f,g)(x) = f(x)g'(x) - f'(x)g(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

Problem 3. Show that $W(\sin x, \cos x) = -1$ for all x.

Problem 4. Show that $W(e^{r_1 x}, e^{r_2 x})$ is never zero if $r_1 \neq r_2$.

Theorem 1. *Let f and g be differentiable functions defined on an open interval I. If f and g are linearly dependent then $W(f,g)(x) = 0$ for all $x \in I$. It follows that if the Wronskian is not zero for at least one point in I then f and g are not linearly dependent, and are thus linearly independent.*

Notice that Theorem 1 holds true for the pairs of functions in the examples above.

Proof. Suppose f and g are linearly dependent on I. Then there exist constants C_1 and C_2, not both zero, such that

$$C_1 f(x) + C_2 g(x) = 0, \quad \text{for all } x \in I.$$

Taking the derivative gives

$$C_1 f'(x) + C_2 g'(x) = 0, \quad \text{for all } x \in I.$$

But, we can view this as a solution to the 2×2 system

$$\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since the system is homogeneous and C_1 and C_2 are not both zero, it follows that the determinant of $\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}$ is zero. Thus, the Wronskian is always zero in I. □

Remark.[2] Let $f(t) = t^2|t|$ and $g(t) = t^3$. These give us an example of a pair of differentiable linearly independent functions for which the Wronskian is always zero. (Check these claims.) Why does this not contradict Theorem 1? However, we will see later (Theorem 5) that if f and g are solutions to $y'' + py' + qy = 0$ then they are linearly independent if and only if their Wronskian is never zero in the relevant interval.

4. The Theory

We now begin our discussion of differential equations.

Theorem 2 (Existence and Uniqueness: 3.2.1 page 146)**.** *Let $p(x)$, $q(x)$, and $\phi(x)$ be continuous functions on an open interval I and let $x_0 \in I$. Then the initial value problem*

$$y'' + py' + qy = \phi \quad \text{with } y(x_0) = \alpha \text{ and } y'(x_0) = \beta,$$

[2] This was problem 28 in section 3.3 of the 6th edition but is no longer included.

has a unique solution for $y(x)$ on I.

Example 1. Solve the initial value problem,

$$y'' + \frac{t}{t^2-1}\,y' + \frac{1}{t-2}\,y = e^t \quad y(0) = 1 \,\&\, y'(0) = 2.$$

Solution. A unique solution is guaranteed to exist on $(-1, 1)$. If we had as initial conditions $y(1.5) = 7$ & $y'(1.5) = 23$ then a unique solution is guaranteed to exist on $(1, 2)$. If we had as initial conditions $y(-13.5) = 6$ & $y'(-13.5) = 10^{23}$ then a unique solution is guaranteed to exist on $(-\infty, -1)$. □

Remark. The proof of Theorem 2 is very difficult and we will not do it. However, you should understand the statement of the theorem and its many implications and applications. It is the most important theorem in Chapter 3. We will, in what follows, prove some special cases of this theorem and we will use it to gain an understanding of the structure of the solution set of a second order linear differential equation. The general idea is this. For any second order linear homogeneous equation ($y'' + py' + qy = 0$) there exists a pair of linearly independent solutions, f and g. Any initial value problem ($y(a) = b, y'(a) = c$) can be solved by a unique *linear combination* of f and g. (A linear combination of two functions f and g is any function that can be expressed as $C_1 f + C_2 g$.)

In sections 3.5 and 3.6 we will show that a solution to a nonhomogeneous problem can be found by adding an extra term to any solution of the corresponding homogeneous problem.

Theorem 3 (Principle of Superposition: 3.2.2 page 147)**.** *If $y = f(x)$ and $y = g(x)$ are solutions of the* homogeneous *differential equation, $y'' + py' + qy = 0$, then so is any linear combination of $f(x)$ and $g(x)$.*

Proof. The proof is very easy. Do it. You may be tested on this. This concept will come up over and over again, both in this course and others. □

Theorem 4. *The unique solution of the initial value problem* $ay'' + by' + cy = 0$ *with* $y(x_0) = \alpha$ *and* $y'(x_0) = \beta$ *has a solution given by some linear combination of one of the these three pairs of functions,* $\{e^{r_1 x}, e^{r_2 x}\}, \{e^{rx}, xe^{rx}\}$ *and* $\{e^{\gamma x}\sin\lambda x, e^{\gamma x}\cos\lambda x\}$.

Proof. We have shown that we can always find two linearly independent solutions to $ay'' + by' + cy = 0$ and that these will be one of the three pairs listed in the theorem. Call them f and g. So, the only question is can we solve the system

$$\begin{bmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

The answer depends on the Wronskian. We know that, because of linear independence, the Wronskian is not always zero. But what if it is zero at $x = x_0$? That would be a problem. However, the reader can check that for the three possible solution pairs $\{e^{r_1 x}, e^{r_2 x}\}, \{e^{rx}, xe^{rx}\}$ and $\{e^{rx}\sin\gamma x, e^{rx}\cos\gamma x\}$ the Wronskian is in fact always nonzero. □

Can we push this idea further?

Theorem 5. *Let* $y = f(x)$ *and* $y = g(x)$ *be linearly independent solutions of* $y'' + p(x)y' + q(x)y = 0$*, where* p *and* q *are continuous on an open interval* I*. Then the Wronskian* $W(f, g)$ *is never zero in* I*.*

Remark: This means that if we can find a pair (f and g) of linearly independent solutions to $y'' + p(x)y' + q(x)y = 0$ then we can solve any initial value problem $y(x_0) = \alpha$ and $y'(x_0) = \beta$ with a linear combination of f and g. We record this as Theorem 6 below.

Easy Proof. Suppose that there is a point $x_0 \in I$ where $W(f,g)(x_0) = 0$. Then consider the initial value problem with $y(x_0) = y'(x_0) = 0$. This leads to the system of equations

$$\begin{bmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But if the Wronskian at x_0, which is just the determinant of the matrix above, is zero, then there are infinitely many values of C_1 and C_2 that work. But this contradicts the uniqueness claim of Theorem 2. □

Your text has a much longer proof. It makes use of *Abel's Formula* which is useful in its own right. You should know Abel's Formula, but the proof below is optional reading.

Book's Proof. STEP 1. We will show that the Wronskian is given by the equation

$$W(f,g)(x) = C \exp\left(-\int p(x)\,dx\right),$$

where C is a constant. (This is Abel's Formula: 3.2.7 page 154.) It follows that W is either always zero or never zero.

We know that

$$f'' + pf' + qf = 0,$$

and

$$g'' + pg' + qg = 0.$$

Multiply the first equation by g and the second one by f. Thus,

$$gf'' + pgf' + qgf = 0,$$

and

$$fg'' + pfg' + qfg = 0.$$

Subtract first equation from the second and simplify to get

$$(fg'' - gf'') + p(fg' - f'g) = 0.$$

Notice that $W = fg' - f'g$ and that $W' = fg'' - gf''$. Thus W must satisfy the differential equation

$$W' + pW = 0.$$

But this we can solve, showing that

$$W(x) = C \exp\left(-\int p(x)\,dx\right),$$

for some C. Next we must show that C is not zero.

STEP 2. Suppose that W is always zero in I. Let x_0 be any point in I. Then the system of equations

$$C_1 f(x_0) + C_2 g(x_0) = 0,$$
$$C_1 f'(x_0) + C_2 g'(x_0) = 0,$$

has a nontrivial solutions for C_1 and C_2 (i.e., they are not both zero). Let $h(x) = C_1 f(x) + C_2 g(x)$. Then $y = h(x)$ is a solution to the initial value problem $y'' + py' + qy = 0$ with $y(x_0) = 0$ and $y'(x_0) = 0$. However, it is clear that $y(x) = 0$ (the zero function) solves this system. Thus, by the uniqueness part of Theorem 2, $h(x)$ is the zero function. This, in turn, means that $C_1 f(x) + C_2 g(x) = 0$ for all x in I. Thus, f and g are linearly dependent on I, contradicting our hypotheses. Thus, W is never zero. □

Theorem 6. *Suppose f and g are linearly independent solutions of*

$$y'' + py' + qy = 0.$$

Then any initial value problem, $y(x_0) = \alpha$ and $y'(x_0) = \beta$, is solved by a linear combination of f and g.

Proof. In light of Theorem 5 this is just the same as the proof of Theorem 4. □

Theorem 7. *If f and g are linearly independent solutions of*

$$y'' + py' + qy = 0,$$

then every other solution can be written as a linear combination of f and g.

Proof. Let h be a solution. Let $x_0 \in I$. Let $\alpha = h(x_0)$ and $\beta = h'(x_0)$. Consider the system of equations

$$C_1 f(x_0) + C_2 g(x_0) = \alpha$$
$$C_1 f'(x_0) + C_2 g'(x_0) = \beta.$$

Since the Wronskian of f and g is never zero it is not zero at $x = x_0$. This means we can find unique values of C_1 and C_2 that solve the 2×2 system. By the uniqueness part of Theorem 2 it follows that $h(x) = C_1 f(x) + C_2 g(x)$ for all $x \in I$. □

Theorem 8. *Every differential equation of the form $y'' + py' + qy = 0$, with p and q continuous on I, has two linearly independent solutions.*

Proof. Let x_0 be in I. Consider the two initial value problems

$$y'' + py' + qy = 0 \quad \text{with} \quad y(x_0) = 1 \quad \text{and} \quad y'(x_0) = 0$$

and

$$y'' + py' + qy = 0 \quad \text{with} \quad y(x_0) = 0 \quad \text{and} \quad y'(x_0) = 1.$$

Let f be a solution of the first and g of the second. Their Wronskian at $x = x_0$ is 1 (check this). Thus, f and g are linearly independent. □

Conclusions: So, by using suitable initial values we can show that there exists a pair of linearly independent solutions to any second order linear homogeneous differential equation. Except for the case of constant coefficients and a few other special cases discussed in the text we do not have a procedure for finding them. In section 3.4 (page 170, "Reduction of Order") we did see that if you know one solution, $y_1(x)$, then you can find another of the form

$y_2(x) = v(x)y_1(x)$. This second solution can be shown to be linearly independent of the first.

5. Implications and applications

Students often wonder what the point is of learning the theory. It can seem rather abstract and remote from practical concerns. In an attempt to mitigate this phenomena, this section shows some useful applications that we would not have suspected without the theoretical insights gleaned above.

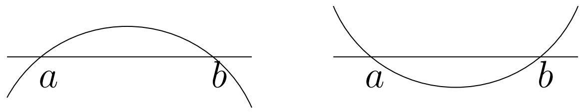
5.1. **The interlacing theorem.**

Theorem 9 (#33 in 3.3.)**.** *Suppose that* y_1 *and* y_2 *are linearly independent solutions of* $y'' + py' + qy = 0$*. Suppose* p *and* q *are continuous on* I*. Suppose* a *and* b *are in* I*,* $a < b$*, and* $y_1(a) = y_1(b) = 0$*. Then there is a point* c *in between* a *and* b*,* $a < c < b$*, such that* $y_2(c) = 0$*. Thus,* y_1 *and* y_2 *are "interlaced".*

Examples: Consider $y'' + y = 0$. The functions $\sin x$ and $\cos x$ are a linearly independent pair of solutions. In between each pair of zeros of $\sin x$, $\cos x$ has a zero, and vice versa. Thus the graphs of these two functions are said to be *interlaced*. This would work for any solution pair of the form $\{e^{\gamma x}\sin\lambda x, e^{\gamma x}\cos\lambda x\}$. [Pick numbers for γ and λ, and graph these two functions to see this.]

We give two proofs. The first might be called an *engineer's proof*. It relies on the graphical intuition. The second, a *mathematicians proof*, is technically correct but does not give one a sense of *why* Theorem 9 should be true.

Pictorial Proof. Let a and b be consecutive zeros of y_1. It is easy to show $y_1'(a) \neq 0$ since otherwise the Wronskian would be zero at a. Likewise $y_1'(b) \neq 0$. Thus, our situation is one of the two cases depicted below.



Notice that in both cases $y_1'(a)$ and $y_1'(b)$ have opposite signs. We know $y_2(a)$ and $y_2(b)$ cannot be zero (otherwise the Wronskian would have a zero). Suppose $y_2(x)$ is never zero in $[a, b]$. Then $y_2(x)$ is either always positive or always negative on $[a, b]$. In particular $y_2(a)$ and $y_2(b)$ have the same sign. Consider $W(y_1, y_2)(t)$ at a and b.

$$W(y_1, y_2)(a) = y_1(a)y_2'(a) - y_1'(a)y_2(a) = -y_1'(a)y_2(a)$$

$$W(y_1, y_2)(b) = y_1(b)y_2'(b) - y_1'(b)y_2(b) = -y_1'(b)y_2(b)$$

Since $y_1'(a)$ and $y_1'(b)$ have opposite signs and $y_2(a)$ and $y_2(b)$ have the same signs, $W(y_1, y_2)(a)$ and $W(y_1, y_2)(b)$ must have opposite signs. Since $W(y_1, y_2)$ is continuous (why?) it must have a zero somewhere in between a and b. But by Theorem 5 the Wronskian is never zero! This contradiction proves that y_2 must have a zero in (a, b). □

Formal Proof. Let a and b be consecutive zeros of y_1. We know the Wronskian is never zero on $[a, b]$. Therefore $y_2(a)$ and $y_2(b)$ are not zero. Assume $y_2(t)$ is never zero for t in (a, b). Let $f(t) = y_1(t)/y_2(t)$. It is differentiable for all $t \in (a, b)$ and continuous on $[a, b]$. Now

$$f'(t) = \frac{y_1'(t)y_2(t) - y_1(t)y_2'(t)}{y_2^2} = \frac{W(t)}{y_2^2(t)}.$$

Therefore $f'(t)$ is never zero in (a, b). But $f(a) = f(b) = 0$. By the Mean Value Theorem there exists a $c \in (a, b)$ such that $f'(c) = 0$. But this is a contradiction. Therefore the assumption that $y_2(t)$ is never zero in (a, b) is false □

5.2. **Impossible pairs Theorems.**

Example 2. Suppose t and t^2 solve $y'' + p(t)y' + q(t)y = 0$. Show that p of q are discontinuous at zero. *Solution.*

$W(t, t^2) = t^2 = 0$ at $t = 0$. This is impossible if $p(t)$ and $q(t)$ are continuous at $t = 0$, by Theorem 5.

Problem 5. We take the last example further. Suppose $y_1(t) = t$ and $y_2(t) = t^2$ solve $y'' + p(t)y' + q(t)y = 0$. Substitute these for y in $y'' + p(t)y' + q(t)y = 0$ to yield two equations with p and q as unknowns. Solve for $p(t)$ and $q(t)$.

In the theorems below assume p and q are continuous on an interval I and that y_1 and y_2 are linearly independent solutions to $y'' + py' + qy = 0$.

Theorem 10 (#38, section 3.2)**.** *The functions y_1 and y_2 cannot both be zero at a point in I.*

Proof. Try to prove this! (We already used this in the proof of Theorem 9.) □

Theorem 11 (#39, section 3.2)**.** *The functions y_1 and y_2 cannot have a max or min at the same point in I.*

Proof. Try to prove this! □

Theorem 12 (#40, section 3.2)**.** *The functions y_1 and y_2 cannot have an inflection point at the same point $a \in I$, unless p and q are both zero at that point.*

Proof. Suppose a is an inflection point of y_1 and y_2. Then $y_1''(a) = y_2''(a) = 0$. Thus $y_1'(a)p(a) + y_1(a)q(a) = 0$ and $y_2'(a)p(a) + y_2(a)q(a) = 0$. We can rewrite this as,

$$\begin{bmatrix} y_1'(a) & y_1(a) \\ y_2'(a) & y_2(a) \end{bmatrix} \begin{bmatrix} p(a) \\ q(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The determinant is equal to $W(y_1, y_2)(a)$ even though the matrix is a little different. Thus the determinant is not zero at a since $W(y_1, y_2)(t)$ is never zero on I. Thus, our matrix equation has the unique solution $p(a) = q(a) = 0$. □