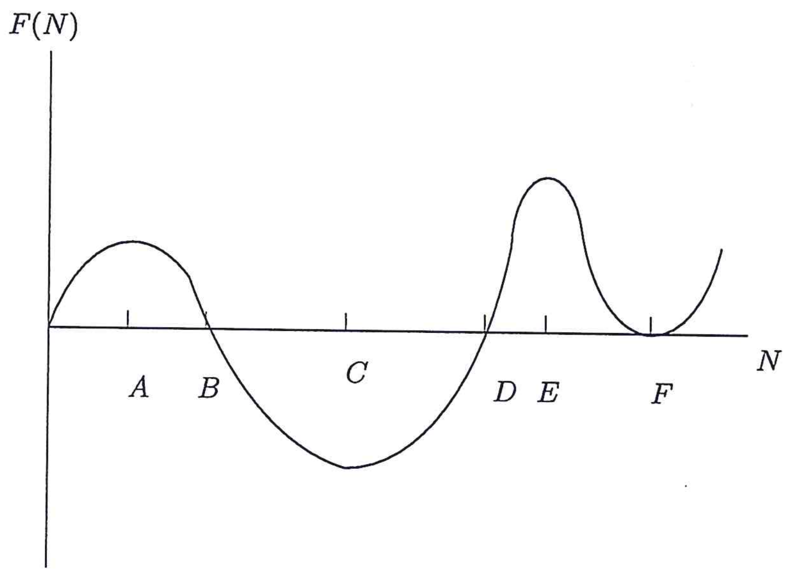
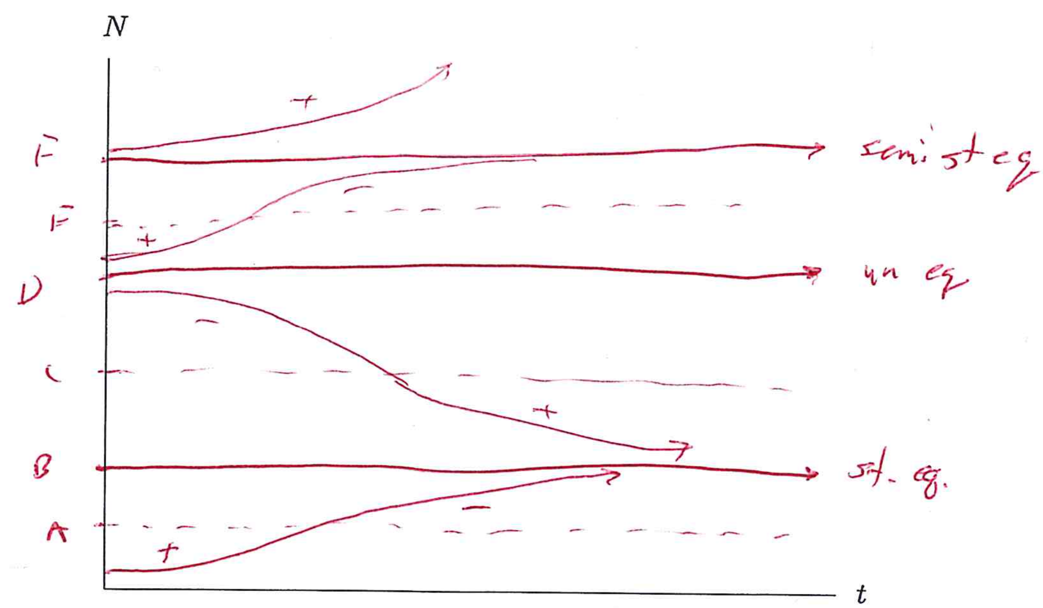
Math 305 Test 1 Spring 2018

Name: Key

ONLY NON GRAPHING CALCULATORS ALLOWED

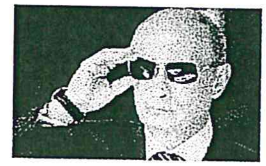
1. [10 points] Suppose $N'(t) = F(N(t))$, where the graph of $F(N)$ is given below. Carefully draw several solution curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the concavity of the solution curves. Assume $N(t)$ and t are non-negative.





2. [10 points] You are in the middle of giving a presentation to a group of potential investors in your new startup company when you realize that Vladimir Putin himself has hacked your laptop and deleted the labels on your direction field slides and inserted an extra slide. You must quickly match the differential equations with their corresponding direction fields. (Each correct match is worth 2 points, each incorrect match is -1 point.)

(a) $y' = x + y$
(b) $y' = y + 1$
(c) $y' = \sin x \cos y$
(d) $y' = y - 3$
(e) $y' = y - 2x$



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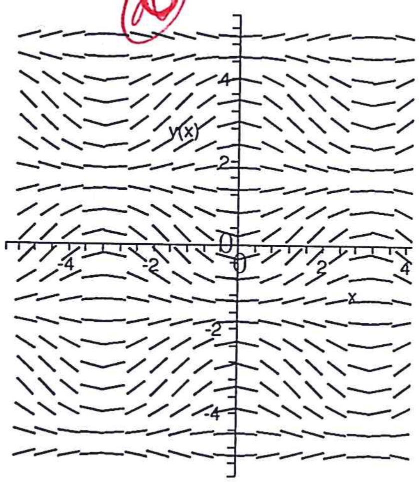
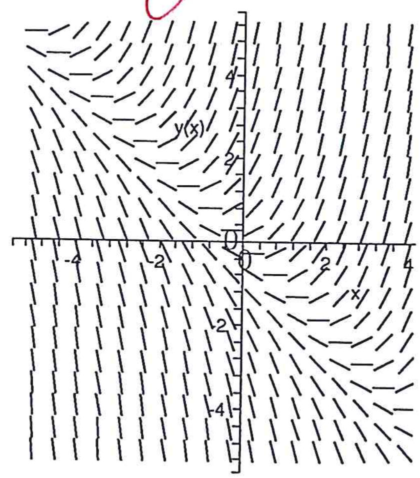
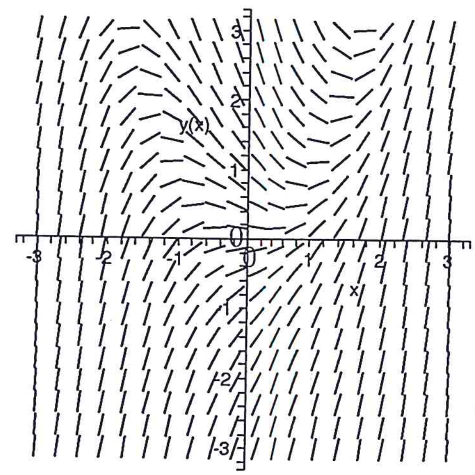


a





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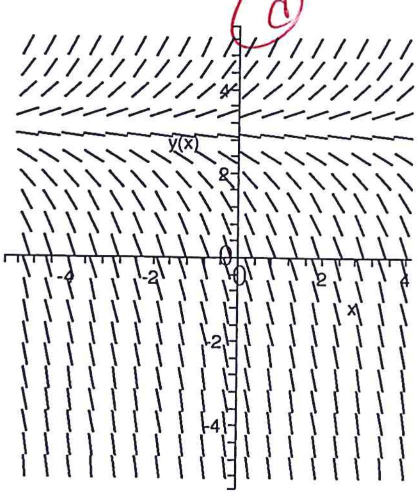

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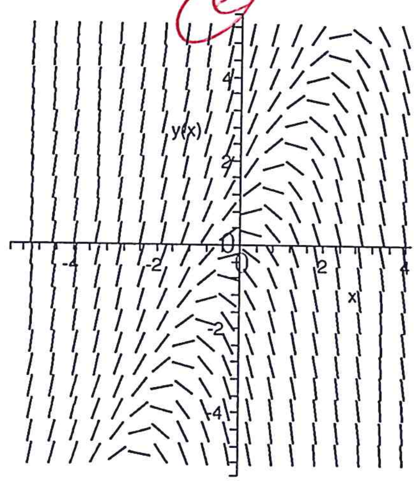

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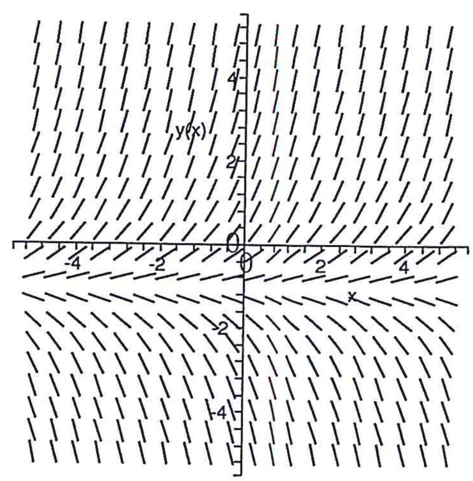


3. [5 points each] Find the general solution to the following.

(a) $\dfrac{dy}{dx} = e^{x-y}$.

$$\int e^y\,dy = \int e^x\,dx$$

$$e^y = e^x + C$$

$$y = \ln(e^x + C)$$

(b) $y' + xy = x$.

Linear

$$\mu = e^{\int x\,dx} = e^{\frac{1}{2}x^2}$$

$$e^{\frac{1}{2}x^2} y' + x e^{\frac{1}{2}x^2} y = x e^{\frac{1}{2}x^2}$$

$$\left(e^{\frac{1}{2}x^2} y\right)' = x e^{\frac{1}{2}x^2}$$

$$e^{\frac{1}{2}x^2} y = \int x e^{\frac{1}{2}x^2}\,dx$$

$$u = \tfrac{1}{2}x^2$$

$$du = x\,dx$$

$$= e^{\frac{1}{2}x^2} + C$$

$$y = 1 + Ce^{-\frac{1}{2}x^2}$$

Separable

$$y' = x - xy = x(1-y)$$

$$\int \frac{1}{1-y}\,dy = \int x\,dx$$

$$-\ln|1-y| = \tfrac{1}{2}x^2 + C$$

$$|1-y| = e^{-\frac{1}{2}x^2 + C}$$

$$= e^C e^{-\frac{1}{2}x^2}$$

$$1 - y = (\pm e^C) e^{-\frac{1}{2}x^2}$$

$$y = 1 + Ce^{-\frac{1}{2}x^2}$$

4. [5 points each] Find the general solution to each of the following.

(a) $\underbrace{2xy+1}_{M}+\underbrace{x^2}_{N}y'=0$ (Check for exactness.)

$M_y = 2x \quad N_x = 2x$ ✓

$\psi = \int M\,dx = \int 2xy+1\,dx = x^2y + x + C_1(y)$

$\psi = \int N\,dx = \int x^2 dy = x^2y + C_2(x)$.

Let $\psi(x,y) = x^2y + x$

solution is $x^2y + x = C$

$y = \dfrac{C-x}{x^2}$

(a) It is also linear.

$x^2y' + 2xy = -1$

$y' + \frac{2}{x}y = \frac{-1}{x^2}$

$\mu = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$x^2y' + 2xy = -1$

$(x^2y)' = -1$

$x^2y = -x + C$

$y = \dfrac{C-x}{x^2}$

(b) $y'' - 2y' - 3y = 0$.

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3 \quad r = -1$

$y = C_1e^{3x} + C_2e^{-x}$.

5. [15 points] Find the general solution to $y' + 3y = y^3$. (Try Bernoulli's method.)

Bernoulli's

$v = y^{1-n} = y^{1-3} = y^{-2}$.

$y = v^{-\frac{1}{2}}$. $y' = -\frac{1}{2} v^{-\frac{3}{2}} v'$

$y^3 = v^{-\frac{3}{2}}$.

Plug in:

$-\frac{1}{2} v^{-\frac{3}{2}} v' + 3 v^{-\frac{1}{2}} = v^{-\frac{3}{2}}$

$v' - 6v = -2$

$\mu = e^{\int -6\,dx} = e^{-6x}$

$e^{-6x} v' - 6 e^{-6x} v = -2 e^{-6x}$

$(e^{-6x} v)' = -2 e^{-6x}$

$e^{-6x} v = \frac{1}{3} e^{-6x} + C$

$v = \frac{1}{3} + C e^{+6x}$ plus sign

$y = v^{-\frac{1}{2}} = \dfrac{\pm 1}{\sqrt{\frac{1}{3} + C e^{6x}}}$

It is separable, but the integration is hard.

$\dfrac{dy}{dx} = y^3 - 3y$

$\int \dfrac{1}{y^3 - 3y}\,dy = \int dx = x + c$

$= \int \dfrac{1}{y(y^2-3)}\,dy = \int \dfrac{a}{y} + \dfrac{b}{y-\sqrt{3}} + \dfrac{c}{y+\sqrt{3}}\,dy$

$= \int \dfrac{-1/3}{y} + \dfrac{1/6}{y-\sqrt{3}} + \dfrac{1/6}{y+\sqrt{3}}\,dy$ (see if you can figure out how got this.)

$= -\frac{1}{3} \ln|y| + \frac{1}{6} \ln|y-\sqrt{3}| + \frac{1}{6} \ln|y+\sqrt{3}| = x + c$

$2 \ln|y| + \ln|y-\sqrt{3}| - \ln|y+\sqrt{3}| = -6x + C$

$\ln\left| \dfrac{y^2}{y^2-3} \right| = -6x + C$

$\dfrac{y^2}{y^2-3} = C e^{-6x}$

$y^2 (1 - C e^{-6x}) = -3 C e^{-6x}$

$y^2 = \dfrac{-3 C e^{-6x}}{1 - C e^{-6x}} = \dfrac{-3}{\frac{1}{C} e^{6x} - 1} = \dfrac{1}{C e^{6x} + \frac{1}{3}}$

$y = \dfrac{\pm 1}{\sqrt{\frac{1}{3} + C e^{6x}}}$

Find the particular solution for $y(0) = \sqrt{3}$.

$x = 0$, $y = \sqrt{3}$. use $+$.

$\sqrt{3} = \dfrac{1}{\sqrt{\frac{1}{3} + C}}$. Let $C = 0$. Thus $y(x) = \sqrt{3}$.

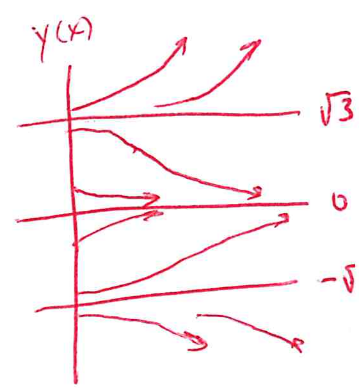
Find the particular solution for $y(0) = 0$.

$x = 0$, $y = 0$.

$0 = \dfrac{1}{\sqrt{\frac{1}{3} + C}}$, has no solution.

But notice $y(x) = 0$ solves the original problem since $y' = 0' = 0$ gives $0 + 3 \cdot 0 = 0^3$.

$\boxed{\text{We "lost" } y = 0 \text{ when we set } v = \frac{1}{y^2}}$



In fact the eq is autonomous: $y' = y^3 - 3y = y(y^2 - 3)$

Let $v = y'$. Then $y'' = v'$

6. [15 points] $y'y'' = 4x$.

Hint: $\int \sqrt{ax^2+b}\,dx = \frac{x}{2}\sqrt{ax^2+b} + \frac{b}{2\sqrt{a}} \ln\left|\sqrt{a}x + \sqrt{ax^2+b}\right| + C$.

$$v v' = 4x$$

$$\int v\,dv = \int 4x\,dx$$

$$\frac{v^2}{2} = 2x^2 + C_1$$

$$v = \pm\sqrt{4x^2 + C_1}$$

$$y = \int v\,dx = \pm \int \sqrt{4x^2+C_1}\,dx =$$

$$\pm\left[\frac{x}{2}\sqrt{4x^2+C_1} + \frac{C_1}{4}\ln\left|2x + \sqrt{4x^2+C_1}\right| + C_2\right]$$

$$y'(0) = v(0) = 2 = \pm\sqrt{4\cdot 0^2 + C_1}.$$

use $+$. $C_1 = 4.$

$$y(0) = 1 = \cancel{\pm} + \left[0 + \frac{4}{4}\ln\left|\sqrt{4}\right| + C_2\right]$$

$$C_2 = 1 - \ln 2.$$

Find the two integration constants if $y(0) = 1$ and $y'(0) = 2$.

I meant to type min. I gave full credit if you used 2 gal/min or 2 gal/hour.

7. [15 points] A tank has 100 gal of freshwater. At $t = 0$ brackish water starts to spill in at 2 gal/hour. Its salinity is α lb/gal. The well mixed water flows out of tank at the same rate. After fifty minutes the salinity of the tank water is measured and found to be 0.25 lb/gal. Find α. (Watch your algebra!)

~~2 gal/hour~~

$$\frac{dQ}{dt} = 2\alpha - 2\frac{Q}{100}$$

$$Q' + \frac{1}{50}Q = 2\alpha$$

$$\mu = e^{\int \frac{1}{50}dt} = e^{\frac{t}{50}}$$

$$e^{t/50}Q' + \frac{1}{50}e^{t/50}Q = 2\alpha e^{t/50}$$

$$(e^{t/50}Q)' = 2\alpha e^{t/50}$$

$$e^{t/50}Q = \frac{2\alpha}{\frac{1}{50}}e^{t/50} + C$$

$$Q(t) = 100\alpha + Ce^{-t/50}$$

$Q(0) = 0 \Rightarrow C = -100\alpha$

2 gal/min

$Q(50) = .25 \times 100 = 25$

$Q(50) = 100\alpha(1 - e^{-1}) = 25$

$$\alpha = \frac{25}{100}\frac{1}{1-e^{-1}} \approx 0.39549$$

2 gal/hour

$50\text{ min} = \frac{5}{6}\text{ hour}$

$Q(\frac{5}{6}) = 25$

$Q(\frac{5}{6}) = 100\alpha(1 - e^{-\frac{1}{60}}) = 25$

$$\alpha = \frac{25}{100}\frac{1}{(1-e^{-1/60})}$$

≈ 15.125