Complex Numbers for Math 305

Complex numbers are used regularly in this course. It is assumed you have worked with complex numbers before, but since they are not used in the Calculus sequence, we review them here. The key idea is to use a symbol i to stand for a supposed $\sqrt{-1}$. The symbol i is called the imaginary unit. Using i it has been proved that every polynomial equation $p(z) = 0$ has a solution of the form $a + ib$, where a and b are real numbers.

The complex number field, denoted $\mathbb{C}$, is the set $\{a+bi \mid a, b \in \mathbb{R}\}$ subject to the rules

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

and

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

Subtraction is defined in the obvious way. Clearly, $\mathbb{C}$ contains the real numbers $\mathbb{R}$ as a subset. Complex numbers of the form bi are said to be purely imaginary. Here are some examples.

- $(3 + 4i) - (7 - 2i) = -4 + 6i$.
- $5i(2 + 3i) = -15 + 10i$.
- $(2 - 6i) + 2i(1 + i) = (2 - 6i) + (-2 + 2i) = -4i$.
- $i^{30} = i^{4\cdot 7+2} = 1^7(i^2) = -1$.

Division is a bit subtle. The expression $w = 1/z$ means $zw = 1$. Thus, if we want to find $1/(3 + 8i)$ we have to solve $(3 + 8i)(c + di) = 1$. This gives $3c - 8d = 1$ and $8c + 3d = 0$. Solving this system of equations gives $c = 3/(3^2 + 8^2)$ and $d = -8/(3^2 + 8^2)$. The general formula is

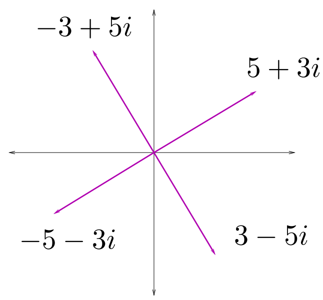
$$\frac{1}{a + bi} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i.$$

Example.

$$\frac{2 + 3i}{5 - i} = \frac{(2 + 3i)(5 + i)}{26} = \tfrac{7}{26} + \tfrac{17}{26}i.$$

Initially, many viewed complex numbers as merely a mathematical curiosity of little practical importance. But, in the 1800s it was discovered that they had interesting geometrical properties which made them useful in physics. We define the plane of complex numbers as having the horizontal axis representing the real numbers and the vertical axis representing the purely imaginary numbers. View the complex numbers as vectors based at the origin. Notice that multiplication by -1 rotates a complex number by

180^o and that multiplication by i rotates a complex number by 90^o counter-clockwise. Now the equation $i^2 = -1$ has a geometric meaning - it models a physically meaningful behavior. Further, multiplication by $i^3 = -i$ rotates a complex number by 270^o counterclockwise or 90^o clockwise, and $i^4 = 1$ rotates a complex number by 360^o which is to say multiplication by 1 does nothing. See the figure below for an example.



We introduce some terminology. The *complex conjugate* of a complex complex number $z = x + yi$ is $\overline{z} = x - yi$. The *magnitude* of a complex number $z = x + yi$ is $|z| = \sqrt{x^2 + y^2}$. Notice that $|z|^2 = z\overline{z}$. The *argument* of a complex number $z = x + yi$ is the angle θ that the vector $\langle x, y\rangle$ makes with the positive real axis. The formula is $\arg(z) = \arctan(y/x)$, up to an additive factor of π depending on the quadrant. Notice that $1/z = \overline{z}/|z|^2$. Here are some examples.

- $|2 + 3i| = \sqrt{4 + 9} = \sqrt{13}$.
- $(2 + 3i)\overline{(2 + 3i)} = (2 + 3i)(2 - 3i) = 4 - 6i + 6i - 9i^2 = 13$.
- $\arg(2 + 3i) = \arctan(3/2) \approx 0.98279\,\text{rad}$.
- $\arg(-2 + 3i) = \arctan(-3/2) + \pi \approx 2.15880\,\text{rad}$.

A complex number $z = x + yi$ can be expressed in polar form as follows. Let $R = |z|$ and $\theta = \arg(z)$. Then

$$z = R(\cos\theta + i\sin\theta) = Re^{i\theta}.$$

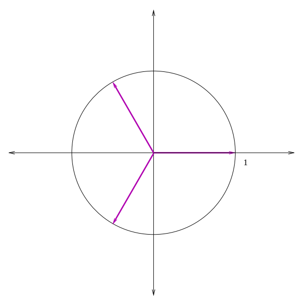
The first equality is just trigonometry. The second equality, called *Euler's formula*, may seem bizarre at first, but it can be derived using the Taylor series expansions of $\cos\theta$, $\sin\theta$ and $e^{i\theta}$. This will be done in class later. The polar form allows us to see the geometric meaning of complex multiplication. Let $z = R_1e^{i\theta}$ and $w = R_2e^{i\phi}$. Then

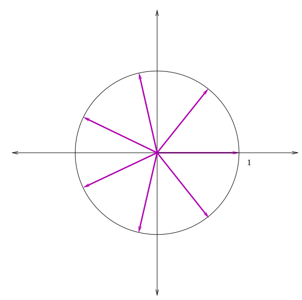
$$zw = R_1R_2e^{i(\theta+\phi)}.$$

In words, the magnitudes multiply while the arguments add.

A complex number z is a *root of unity* if $z^n = 1$ for some positive integer n. For $n = 2$ there are two roots of unity, 1 and -1. For $n = 4$ there are four roots of unity, 1, i, -1, and $-i$. What about the roots of unity for $n = 3$? We can find these by observing that $z^3 = 1$ can be written in polar form $z^3 = (Re^{i\theta})^3 = R^3 e^{i3\theta} = 1$. Set $R = 1$. Then we need for 3θ to be a multiple of 2π. This gives $\theta = 0, \pm 2\pi/3, \pm 4\pi/3, \pm 6\pi/3, \ldots$. But, only three of these are unique, 0, $2\pi/3$, and $4\pi/3$. These give 1, $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$, and $\frac{-1}{2} - \frac{\sqrt{3}}{2}i$ as the three solutions to $z^3 = 1$.

For $n = 7$ there are seven roots of unity. The angles are 0, $2\pi/7$, $4\pi/7$, $6\pi/7$, $8\pi/7$, $10\pi/7$, $12\pi/7$. Thus, the seven solutions of $z^7 = 1$ are $z = \cos(2k\pi/7) + i\sin(2k\pi/7)$ for $k = 0, 1, 2, 3, 4, 5, 6$. See the figures below.





Problem. Solve $z^3 + 1 = 0$. This is similar to the roots of unity problems, but we need to solve $z^3 = -1$. Again let $z = Re^{i\theta}$ be the polar form. Set $R = 1$. Now we need for 3θ to be an odd multiple of π. Thus, $\theta = (2k+1)\pi/3$. This gives $\pi/3$, π, and $5\pi/3$ as the possible arguments. Thus the solution set for z is

$$\left\{\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, -1, \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i\right\}$$

Problem. Solve $z^4 + 16 = 0$. Now we need $|z^4| = |-16|$. So, we let $z = 2e^{i\theta}$. We need for $4\theta = (2k+1)\pi$. This gives $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ as values for θ. Thus the solution set for z is

$$\{2\cos((2k+1)\pi/4) + 2i\sin((2k+1)\pi/4) \,|\, k = 0, 1, 2, 3\}$$

$$= \left\{\sqrt{2} + i\sqrt{2}, -\sqrt{2} + i\sqrt{2}, -\sqrt{2} - i\sqrt{2}, \sqrt{2} - i\sqrt{2}\right\}.$$

Exercise. Find all the solutions of $(z^2 + 4)(z^5 - 243) = 0$. Plot them in the complex plane.