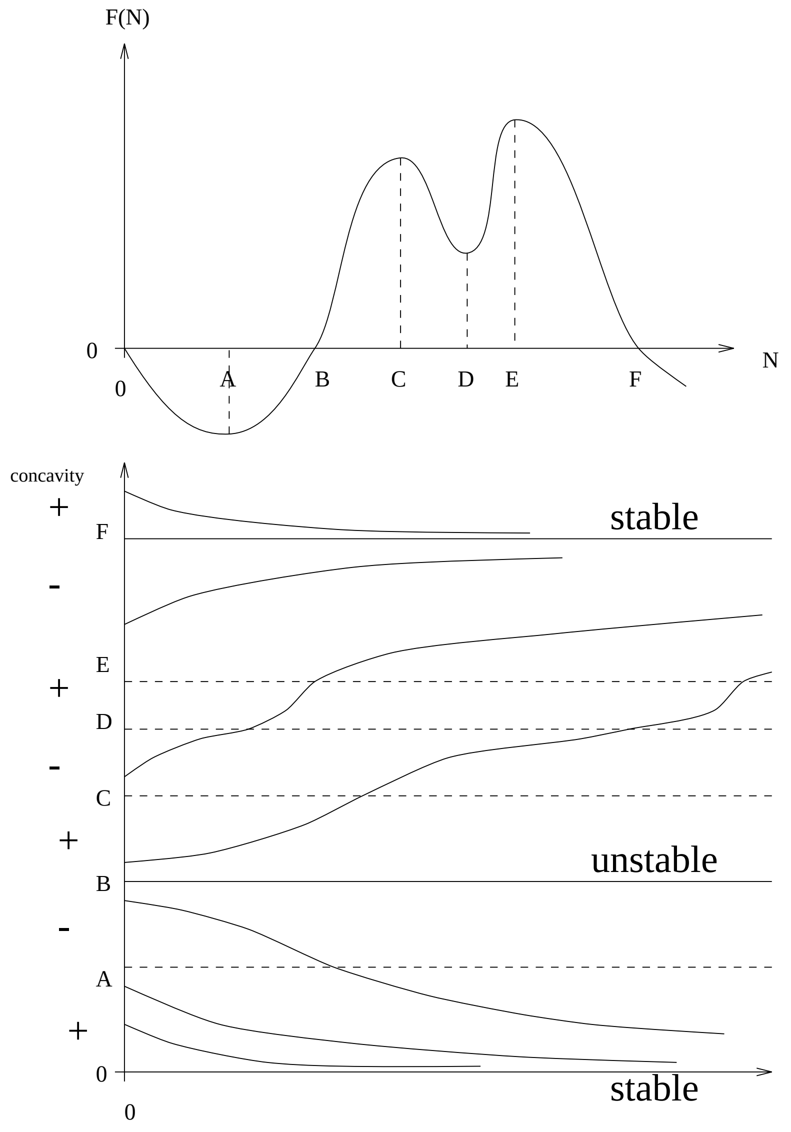
Solutions to Practice Test 1: Math 305, Spring Semester

1. Suppose $N'(t) = F(N(t))$, where the graph of $F(N)$ is given below. Carefully draw the integral curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the initial concavity of the solution curves. Assume $N(t)$ and t are nonnegative.



2. Find the general solution of the differential equation $y' + y \sin x = \sin x$.

Let $\mu(x) = e^{\int \sin x \, dx} = e^{-\cos x}$. Then we get

$$e^{-\cos x} y' + \sin x e^{-\cos x} y = \sin x e^{-\cos x}$$

Hence,

$$(e^{-\cos x} y)' = \sin x e^{-\cos x}.$$

The integral on the right hand side is done by letting $u = -\cos x$. Now we have

$$e^{-\cos x} y = e^{-\cos x} + C.$$

It is easy to solve for y.

$$y = 1 + Ce^{\cos x}$$

3. Find the general solution to $2 \sin y \cos x \, dx + \cos y \sin x \, dy = 0$. You do not have to solve for y.

The equation is separable. We integrate.

$$2 \int \cot x \, dx + \int \cot y \, dy = 0.$$

This gives,

$$2 \ln |\sin x| + \ln |\sin y| = C.$$

Combining the logs and exponenting gives

$$\sin^2 x \sin y = C.$$

4. Solve the initial value problem $(1-y-e^{2x})\,dx+dy=0$, with $y(0)=4$. You must solve for y as a function of x. Hint: The equation can be made exact with an integrating factor of the form $\mu(x)$.

We get $M_y=-1$ and $N_x=0$. Thus,

$$\mu(x)=e^{\int \frac{-1-0}{1}\,dx}=e^{-x}.$$

Multiplying through by μ gives,

$$(e^{-x}-ye^{-x}-e^{x})\,dx+e^{-x}\,dy=0$$

Hence,

$$\psi(x,y)=\int M\,dx=\int e^{-x}-ye^{-x}-e^{x}\,dx=-e^{-x}+ye^{-x}-e^{x}+C_1(y).$$

And,

$$\psi(x,y)=\int N\,dy=\int e^{-x}\,dy=ye^{-x}+C_2(x).$$

Thus, $\psi=-e^{x}+(y-1)e^{-x}=C$ is the general solution. Solving for y gives,

$$y=Ce^{x}+e^{2x}+1.$$

5. A body of mass m falls from rest in a medium offering resistance proportional to the square of the velocity. Find the relation between the velocity v and the time t. Find the limiting velocity, v_l.

Start with $v'=g-\frac{k}{m}v^2$. It's separable. We get,

$$\int \frac{dv}{g-\frac{k}{m}v^2}=\int dt.$$

Multiply both sides by g, and factor the denominator in the dv integral. This gives,

$$\int \frac{dv}{\left(1+\sqrt{\frac{k}{mg}}v\right)\left(1-\sqrt{\frac{k}{mg}}v\right)}=\int g\,dt.$$

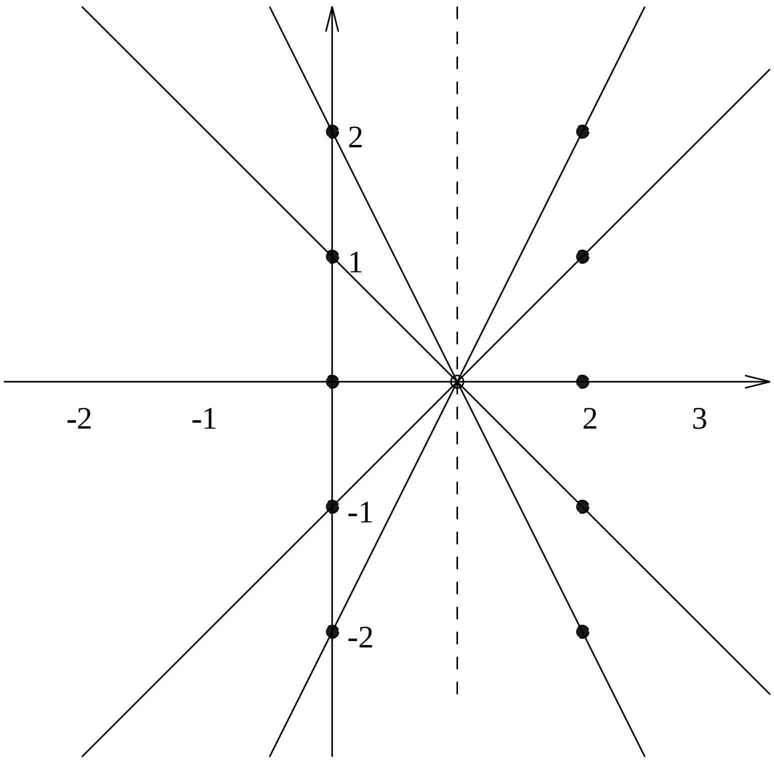
Using the 3rd integral formula from below, we get,

$$\frac{1}{2\sqrt{\frac{k}{mg}}}\ln\left(\frac{\left(1+\sqrt{\frac{k}{mg}}v\right)}{\left(1-\sqrt{\frac{k}{mg}}v\right)}\right)=gt+C.$$

Hence, $v_l=\sqrt{\frac{mg}{k}}$, (why?), and since, $v_0=0$, we get, $C=0$.

6. Consider the equation
$$y' = \frac{y}{x-1}.$$
Find the general solution. Draw the integral "curves" for the following initial values: $y(0) = \pm 2, \pm 1, 0$, and $y(2) = \pm 2, \pm 1, 0$. Hint: be careful with your absolute value signs.

The general solution is $y = C|x-1|$. The graph is below:



Integrals:

$\int \cot x \, dx = \ln|\sin x| + C$

$\int 1/(x^2+1) \, dx = \tan^{-1} x + C$

$\int 1/((x+a)(x+b)) \, dx = (1/(a-b)) \ln|(x+a)/(x+b)| + C$