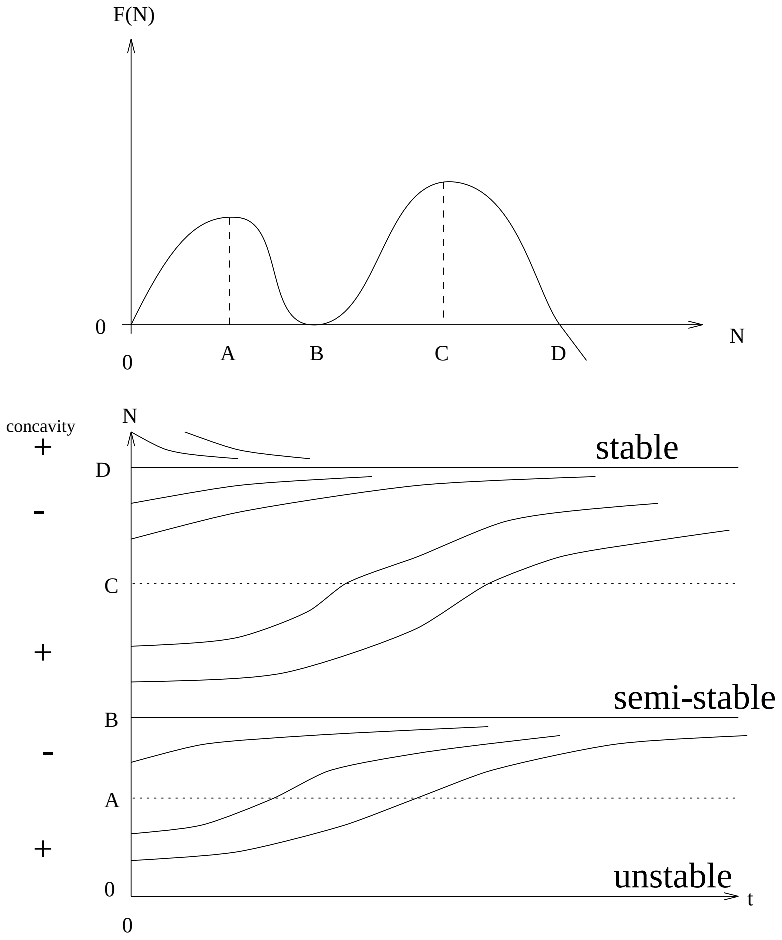
Test 1, Solutions, Math 305, Spring Semester, 1997

1. Suppose $N'(t) = F(N(t))$, where the graph of $F(N)$ is given below. Carefully draw the integral curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the initial concavity of the solution curves. Assume $N(t)$ and t are nonnegative.



2. Solve the initial value problem, $(x^2+1)y' + 3xy = 6x$, with initial condition $y(0) = 2$.

$$y(x) = 2$$

3. Find the general solution of the differential equation $(x^2+y)y' + 2xy = 6x$. Hint: check for exactness. (You need not solve for y.)

Rewrite as

$$\underbrace{2x(y-3)}_{M}\ dx + \underbrace{(x^2+y)}_{N}\ dy = 0.$$

Thus,

$$M_y = 2x \qquad \text{and} \qquad N_x = 2x.$$

Hence the system is exact. Now,

$$\psi(x,y) = \int \psi_x\, dx = \int M\, dx = \int 2x(y-3)\, dx = x^2(y-3) + C_1(y).$$

But also,

$$\psi(x,y) = \int \psi_y\, dy = \int N\, dy = \int x^2 + y\, dy = x^2y + \frac{1}{2}y^2 + C_2(x).$$

Letting

$$C_1(y) = \frac{1}{2}y^2 \qquad \text{and} \qquad C_2(x) = -3x^2,$$

we get,

$$\psi(x,y) = x^2y - 3x^2 + \frac{1}{2}y^2.$$

Hence, the general solution is

$$x^2(y-3) + \frac{1}{2}y^2 = C.$$

4. Find the general solution of the equation below. (You need not solve for y.)

$$(x^2y + yx - y)dx + x^2(y-2)dy = 0$$

Rewrite as

$$\frac{x^2+x-1}{x^2}dx + \frac{y-2}{y}dy = 0,$$

or

$$(1 + \frac{1}{x} - \frac{1}{x^2})dx + (1 - \frac{2}{y})dy = 0.$$

The equation is now separable and we merely have to integrate. This gives us,

$$x + \ln|x| + \frac{1}{x} + y - \ln y^2 + C,$$

as the general solution.

5. The equation below is homogeneous (in the sense of section 2.9). Find its general solution. (You need not solve for y.)

$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$$

Rewrite as

$$\frac{dy}{dx} = \frac{1}{2}\frac{x}{y} + \frac{3}{2}\frac{y}{x} = \frac{1}{2}\left(\left(\frac{y}{x}\right)^{-1} + 3\left(\frac{y}{x}\right)\right) = \frac{1}{2}\left(\frac{1}{v} + 3v\right).$$

Then

$$\frac{dxv}{dx} = v + x\frac{dv}{dx} = \frac{1}{2}\left(\frac{1}{v} + 3v\right).$$

Separating the variables gives us the integrals,

$$\int \frac{2\,dv}{\left(\frac{1}{v} + v\right)} = \int \frac{dx}{x} = \ln|x| + C = \ln|Cx|.$$

The dv integral is done as follows. Multiply top and bottom by v. Then we get

$$\int \frac{2v\,dv}{1+v^2}$$

Let $u = 1 + v^2$. Then $du = 2vdv$, and the integral is just

$$\int \frac{du}{u} = \ln|u| = \ln(1+v^2).$$

Finally substituting $v = y/x$ and simplifying gives

$$y^2 = (C|x| - 1)x^2.$$

But this solution can only be valid for $|x| \leq C$.

6. **BONUS PROBLEM.** The number of algae cells in a tank of water grows according to

$$\begin{aligned} A'(t) &= .2(1 - \frac{A(t)}{100})A(t) \qquad &\text{light on, and} \\ A'(t) &= -.2A(t) \qquad &\text{light off.} \end{aligned}$$

In words, the carrying capacity drops from 100 (billion cells) to zero when the light goes out. At $t = 0$ you start a ten (10) hour experimental run with $A(0) = 25$ and plan to keep the light on. When you come back at $t = 10$ you discover that the light bulb has burned out. You measure $A(10)$ to be 15. What time did the light bulb burn out?

Hints: You can use your graphing calculator, but make sure your answer in correct to at least 4 decimal places. The integral below will be helpful:

$$\int \frac{dx}{x(a+bx)} = \frac{1}{a}\ln\left(\frac{x}{a+bx}\right) + C$$

Solve the "lights on" equation with $A(0) = 25$, to get

$$A(t) = \frac{100}{1+3e^{-.2t}} \qquad \text{for } 0 \le t \le \text{burn out}$$

Solve the "lights of" equation with $A(10) = 15$, to get

$$A(t) = 15e^{2-.2t} \qquad \text{for burn out} \le t \le 10$$

The time t where they intersect is the "burn out time". this can be done by zooming in on the graph (hard to get this to be very accurate), using an equation solver, or algebraically. For the latter, one gets a quadratic equation in $e^{-.2t}$, which can be solved with the quadratic formula. Then it is easy to get t. The answer is

$$t = 4.50082625\ldots \text{ hours,}$$

or about 4 hours and 30 minutes.