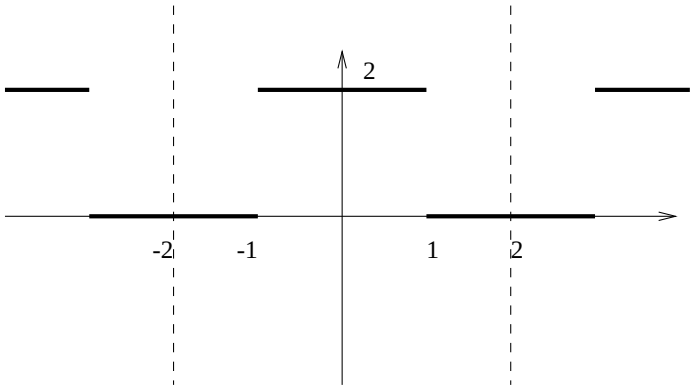
Final Exam **Math 305** **Spring Semester, 1998**

Each problem is worth 25 points.
NO CALCULATORS

I commend you not to learn your sciences from books unaided, even though you may trust your ability to understand. Resort to professors for each science you seek to acquire; and should your professor be limited in his knowledge take all that he can offer, until you find another more accomplished than he. – 'Abd al-Latif, Baghdad, c. 1200.

1. Let $f(x)$ be a periodic function defined by the graph below. Find its Fourier series.



2. Find the general solution of $y'' + 2y' + y = \cos(n\pi x/2)$ where n is any integer.

3. [Hard] The motion of a certain spring-mass systems is governed by the differential equation $u'' + 2u' + u = F(t)$, where $F(t)$ is the function shown in problem 1. If $u(0) = u'(0) = 0$ solve for $u(t)$. Note the similarity with problem 2.

   Hint: The following infinite series may appear in your calculations. Let $C_n = (4/(n\pi))\sin(n\pi/2)$, $A_n = \frac{16-4n^2\pi^2}{16+8n^2\pi^2+n^4\pi^4}$, and $B_n = \frac{16n\pi}{16+8n^2\pi^2+n^4\pi^4}$. Then let

$$\alpha = \sum_{n=1}^{\infty} C_n A_n \approx -.1416086212$$

   and

$$\beta = \sum_{n=1}^{\infty} C_n B_n(n\pi/2) \approx .4935543476.$$

4. For a certian damped mass-spring system the quasi-frequencey is 3. The mass is 1 kg. The maximams decrease by 30% as one goes from peak to peak. What is γ, the damping coeffiecent?

5. Find the series solution to $x^{-1}y'' + (x-1)y' + 3xy = 0$, centered about $x_0 = 0$.

6. a. Draw the direction field of $y_1' = \dfrac{3 - y_1}{2}$. Draw some solution curves.

   b. Draw the direction field of $y_2' = \left(\dfrac{3 - y_2}{2}\right) x$. Draw some solution curves.

   c. Find $\lim_{x\to\infty} y_1(x)$ and $\lim_{x\to\infty} y_2(x)$. Which converges faster? EXPLAIN.

   d. In each case, what happens as $x \to -\infty$?

7. Solve each of the following differential equations.

   a. $\dfrac{dy}{dx} = -\dfrac{2xy + y^2}{x^2 + 2xy}$. DO NOT SOLVE FOR y. Hint: check for exactness.

   b. $(e^x + 1)\dfrac{dy}{dx} = y - ye^x$. Solve for y. Hint: Multiply both sides by $e^{-x/2}$. The integration will be easier.

   c. $xy' = y + xe^{(y/x)}$. Assume $x > 0$. Solve for y. Hint: Let $v = y/x$.

   d. $\dfrac{dy}{dx} + y = \dfrac{1}{1 + e^x}$. Solve for y.

8. A body of mass m falls from rest in a medium offering resistance proportional to the square of the velocity. Find the relation between the velocity v and the time t. Find the limiting velocity, v_l.

   Hint:

$$\int \frac{dx}{a^2 - b^2x^2} = \frac{1}{2ab} \ln\left|\frac{a + bx}{a - bx}\right| + C.$$

9. **BONUS PROBLEM:** Consider a metal ring. Let $u(\theta, t)$ be the temperature at time t at the point θ on the ring. Suppose that at time $t = 0$ we have an initial temperature distribution of $u(\theta, 0) = f(\theta)$. Derive a method for solving such a problem for $u(\theta, t)$.