Math 305 **Final Exam** **Scientific Calculators Allowed**

Print Name: ______________________ Print Code (if you want your grade posted): ______________________

1. [20 points] Consider the differential equation $y + (2x - ye^y)\dfrac{dy}{dx} = 0$ (*).

   (a) [3 points] Show that (*) is not exact.

   (b) [3 points] Show that when we multiply (*) by $\mu = y$ we get an exact differential equation.

   (c) [10 points] Find the general solution to (*) using the result of (b). (Do not try to solve for y; just leave your answer in the form $\psi(x, y) = C$.) Hint: $\int u^2 e^u \, du = (u^2 - 2u + 2)e^u + C$.

   (d) [2 points] Assume $y(0) = 1$ is given. Find C.

   (e) [2 points] Prove that for this initial condition y can never be 0.

2. [20 points] Solve the initial value problem,

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \qquad y(1) = 0.$$

   Express y as a function of x. You may assume $x > 0$. Hint: $\int \frac{1}{1+u^2} du = arctan(u) + C$.

3. [20 points]

   (a) [15 points] Solve the differential equation,

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t \qquad y(0) = a.$$

   Express y as a function of t and a.

(b) [5 points] For what value of a will y be a linear function of t ?

4. [20 points; 5 points each] Consider a cylindrical tank of water with radius 5 ft and height 10 ft. Water flows in at a constant rate of 1 cubic foot per minute.

   In the bottom of the tank there is a 3 inch by 3 inch square hole that water drains out from. The flow rate out is not constant but is determined by Torricelli's theorem in hydrodynamics to be $\alpha a\sqrt{2gh}$, where h is the current height of the water, a is the area of the hole, $g = 32ft/s^2$ is the acceleration due to gravity, and α is a physical constant. Assume $\alpha = 1$.

   Warning: check units, minutes vs seconds in g.

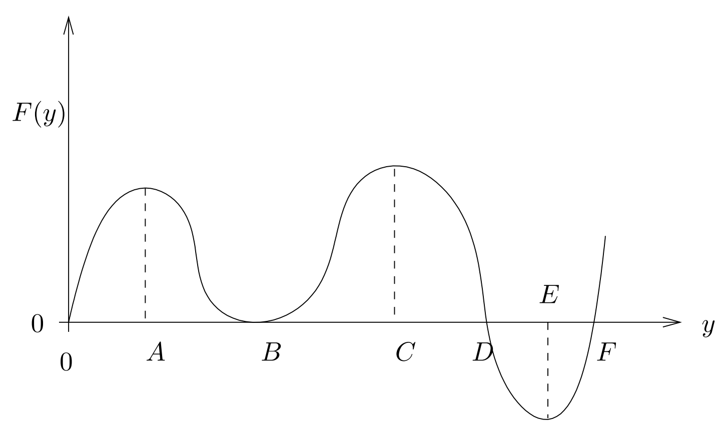
   (a) Set up a differential equation for the volume V of the water at time t. Hint: first find the rate out as a function of V.

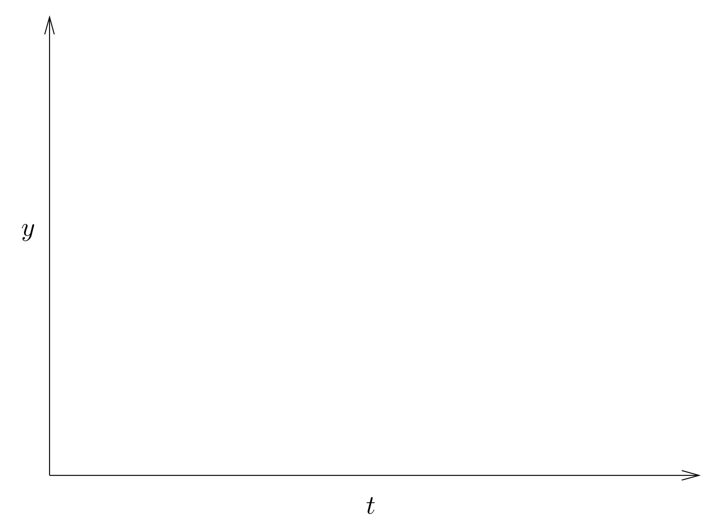
   (b) Without solving the equation, what is the equilibrium solution?

   (c) Explain why it is asymptotically stable. You can draw pictures, but give your explanation in complete sentences.

   (d) Compute the height the water will tend toward as $t \to \infty$.

5. [20 points] Suppose $y'(t) = F(y(t))$, where the graph of $F(y)$ is given below. Carefully draw the integral curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the initial concavity of the solution curves. Assume $y(t)$ and t are nonnegative.





6. [20 points] Let $f(x)$ be an even function and let $g(x)$ be an odd function. For each combination below determine whether it is necessarily even or odd. If the combination is even or odd, give a short proof of this. If the combination is not necessarily even or odd, give an example illustrating this.

   (a) $f(g(x))$

   (b) $g(g(x))$

   (c) $f(x) \cdot g(x)$

   (d) $f(x) \cdot f(x)$

   (e) $f(x) + g(x)$

   (Bonus) Give an example of a function that is both even and odd.

7. [20 points] Assume m and n are positive integers.

   (a) [8 points] Prove that $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0$ if $n \neq m$.

   (b) [6 points] Prove that $\displaystyle\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$, always.

   (c) [6 points] Prove that $\displaystyle\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \pi$, always.

   Hint: These trig identities may be helpful.

   $\sin A \cos B = [\sin(A - B) + \sin(A + B)]/2$

   $\sin A \sin B = [\cos(A - B) - \cos(A + B)]/2$

   $\cos A \cos B = [\cos(A - B) + \cos(A + B)]/2$

8. [20 points] Use the method of separation of variables to replace the partial differential equation below with two ordinary differential equations.

$$u_{xx} + 2u_{xt} - u_t = 0$$

9. [20 points] Let $f(x) = 2 + 7\sin(3x) + 2\cos(2x)$. What is the Fourier Series of $f(x)$ using $L = \pi$? Shown your work or explain your reasoning.

10. [20 points] Consider a metal bar 50 inches long. One end is to be held at 10^o F, the other at 50^o F. Initially the bar is 20^o F.

    (a) [5 points] Write down the partial differential equation with the initial condition and boundary values.

    (b) [5 points] Let $u(x,t)$ represent the solution. What function of x will $u(x,t)$ converge to as $t \to \infty$?

    (c) [10 points] Write out the solution to your equations in (a), but do not evaluate the integrals.