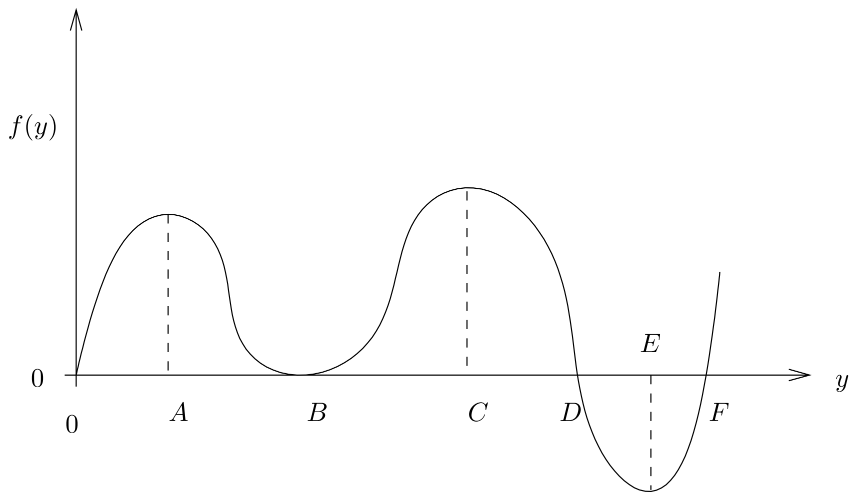
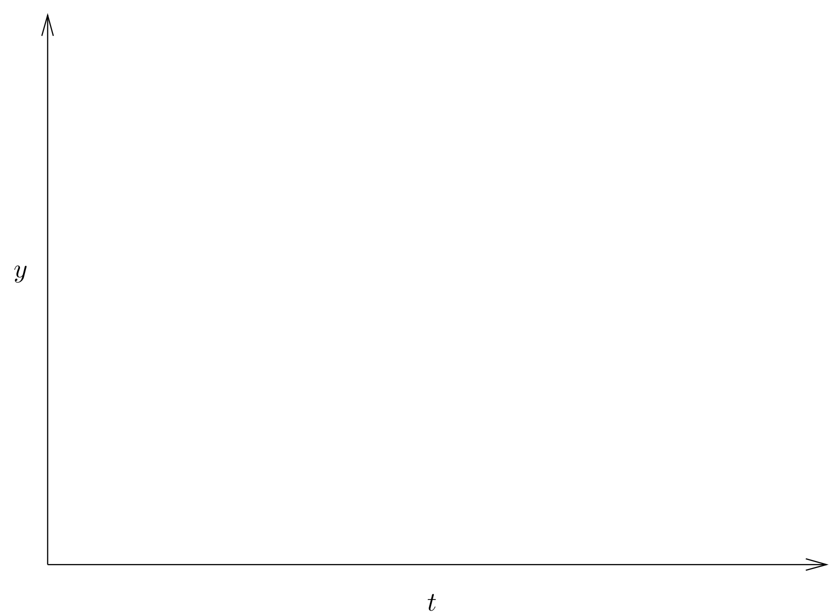
Fall 2004 **Final Exam Part I** **Math 305**

Only Scientific Calculators Allowed

Print Name: ______________________ Code name if you want your grade posted: ______________________

1. [20 points] Suppose $y'(t) = F(y(t))$, where the graph of $F(y)$ is given below. Carefully draw the integral curves for this equation. What are the equilibrium solutions? What are their stability types? Describe the initial concavity of the solution curves. Assume $y(t)$ and t are nonnegative.





2. [20 points] A tank initially contains 120 liters of pure water. A mixture containing a concentration of γ grams/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

3. [20 points] Find the general solutions.

   (a) $y' - 2y = 4 - t$.

   (b) $(2x + 3) + (2y - 2)y' = 0$.

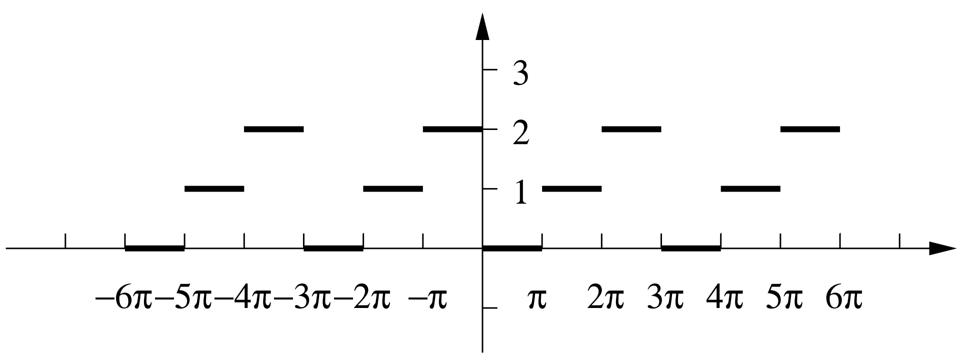
4. [20 points] Find the general solutions.

   (a) $y'' + 4y = 3\cos(2t)$

   (b) $y''' - 3y'' + 3y' - y = 0$.

5. [20 points] Let $y'' + (x - 1)y' + 3y = 0$. Use the series method, centered about $x_0 = 1$, to find the general solution. You must find a recursive formula for a_n. Compute the first four terms if $y(1) = 1$ and $y'(1) = 2$.

6. [20 points] Let $f(x)$ be a periodic function defined by the graph below.



   a. Find a_0.

   b. Find b_5.

7. [5+15 points]

   (a) Show that the Wave Equation is linear.

   (b) Apply the separation of variables method to the Wave Equation and derive the two resulting ODEs.

8. [20 points + 5 bonus points] Let $f(x)$ be an even function and let $g(x)$ be an odd function.

   (a) Let $p(x) = f(x)g(x)$. Is $p(x)$ even or odd? Prove your claim.

   (b) Let $p(x) = g(x)g(x)$. Is $p(x)$ even or odd? Prove your claim.

   (c) Let $p(x) = f(f(x)g(x))$. Is $p(x)$ even or odd? Prove your claim.

   (d) Let $p(x) = f(3x)g(2x) + xf(x)$. Is $p(x)$ even or odd? Prove your claim.

   (e: BONUS) Let $p(x) = f'(x)$. Is $p(x)$ even or odd? Justify your claim.

9. [20 points] The two dimensional Heat Equation is

$$\alpha^2(U_{xx} + U_{yy}) = U_t.$$

Suppose $U(x, y, t) = X(x)Y(y)T(t)$. Show that X, Y and T must satisfy the ordinary differential equations below.

$$T' + \sigma_1\alpha^2 T = 0$$

$$X'' + \sigma_2 X = 0,$$

$$Y'' + (\sigma_1 + \sigma_2)Y = 0$$

10. [20 points] Find the solution of the heat conduction problem

$$\begin{aligned} 100u_{xx} &= u_t, && 0 < x < 1, && t > 0 \\ u(0,t) &= 0, && u(1,t) = 0 && t > 0 \\ u(x,0) = f(x) &= \sin(2\pi x) - 2\sin(5\pi x), && && 0 \le x \le 1. \end{aligned}$$

Find the temperature at $x = 0.5$ when $t = 1.0$.

Hint: The Fouries Sine Series of $f(x)$ is obvious.

11. [20 BONUS points] Develop the theory needed to apply the Heat Equation to a metal ring.

The Heat Equation Summary of Solutions

The One-dimensional Heat equation is

$$\alpha^2 U_{xx} = U_t,$$

where $U(x,t)$ is a function of length x and time t. We assume the initial temperature distribution is given by a function $f(x)$. That is,

$$U(x,0) = f(x)$$

We have considered the Heat Equation with three different boundary (end point) conditions.

1. $U(0,t) = U(L,t) = 0$ for $t > 0$.
2. $U(0,t) = T_1$ and $U(L,t) = T_2$ for $t > 0$.
3. $U_x(0,t) = U_x(L,t) = 0$ for $t > 0$

We arrived at the following respective solutions.

1. $U(x,t) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\alpha^2\pi^2 t}{L^2}}$, where $b_n = \displaystyle\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$.
2. Let $v(x) = \dfrac{T_2 - T_1}{L}x + T_1$. Then $U(x,t) = v(x) + \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\alpha^2\pi^2 t}{L^2}}$, where $b_n = \frac{2}{L}\int_0^L (f(x) - v(x))\sin\left(\frac{n\pi x}{L}\right)\,dx$.
3. $U(x,t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\alpha^2\pi^2 t}{L^2}}$, where $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$.