

Name: \_\_\_\_\_ Section time: \_\_\_\_\_

**SCIENTIFIC CALCULATORS ALLOWED**

1. [20 points] Find the general solution to each of the following.

a.  $2y' = e^{t/3} - y$ .

b.  $(y \cos x) + (\sin x)y' = 0$ .

c.  $y'' + 2y' - 15y = 0$ .

d.  $y'' + 2y' + 2y = 0$ .

2. [20 points] Solve the initial value problem

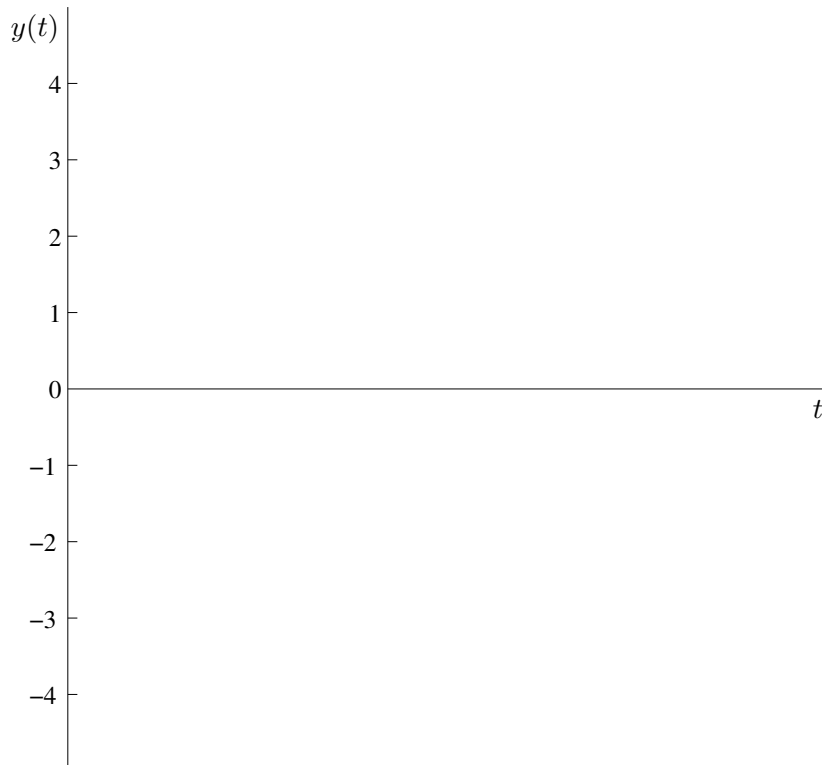
$$t^2 y'' + 2ty' = 1 \quad (t > 0), \quad y(1) = 3, \quad y'(1) = 2.$$

3. [20 points] Let
- $F(y) = \begin{cases} y \ln\left(\frac{|y|}{2}\right) (y^3 - 27) & y \neq 0 \\ 0 & y = 0. \end{cases}$

Now consider the autonomous differential equation

$$\frac{dy}{dt} = F(y).$$

Find the equilibrium solutions and determine their stability types. Draw the phase portrait sketch several solution curves. Indicate (roughly) their concavity. Hint: Remember  $\ln 1 = 0$ .



Bonus: Show that  $F(y)$  is in fact continuous at  $y = 0$ . Do this by studying the limit of  $F(y)$  as  $y \rightarrow 0$ . You'll need L'Hopital's Rule.

4. [20 points] Consider the equation  $y' = 1 + t^2 - 2ty + y^2$ . This is an example of a *Riccatti equation*.

a. Show that  $y(t) = t$  is a solution.

b. Let  $y = t + \frac{1}{v(t)}$ . Substitute this into the given Riccatti equation and derive a new differential equation in  $v$ . Solve this equation for  $v(t)$ . (There will be one arbitrary constant in your solution for  $v(t)$ .) Thus you have found a family of additional solutions to the original Riccatti equation!

5. [20 points] Let

$$f(t) = \begin{cases} t & t \in [0, 5\pi] \\ 2 & t > 5\pi. \end{cases}$$

Find a differentiable solution to the initial value problem

$$y'' + y = f(t) \quad (t \geq 0),$$

$$y(0) = 0 \text{ and } y'(0) = 1.$$

6. [20 points] Find the general power series solution to

$$xy'' + (x + 2)y' + 3y = 0$$

centered about  $x = 0$ . Find a recursive formula for  $a_n$ .

Actually this problem is ill posed. The radius of convergence is 0.

7. [20 points] Consider  $y'' + e^x y' - (3 \sin x)y = 0$ . Suppose  $y(0) = 1$  and  $y'(0) = 2$ . Find the first five terms of the power series solution. What can you say about its radius of convergence?
8. [20 points] a. Graph the square wave given by

$$f(x) = \begin{cases} 1 & x \in [n, n + 1) \text{ for } n \text{ even} \\ -1 & x \in [n, n + 1) \text{ for } n \text{ odd,} \end{cases}$$

over  $x \in [-3, 3]$ .

b. Find the Fourier Series of  $f(x)$ .

9. [20 points] The heat equation does not account for the fact that heat may dissipate from the material to the surrounding environment. Let  $A$  the ambient temperature that a metal rod is in and assume it is constant.

We know from Newton's Law of Cooling that heat flow is proportional to the difference between  $u(x, t)$  and  $A$ . This gives the PDE

$$u_t = au_{xx} - \gamma(u - A).$$

Convince yourself that this equation is not separable. But make the substitution  $w(x, t) = u(x, t) - A$  and derive a PDE in  $w$ . Show that this equation is separable.

10. [20 points] Wave equation can be applied to traveling waves. Here the string is infinitely long. So, there are no boundary conditions. We will let  $f(x)$  be some initial configuration and assume the initial speed of motion is zero. Thus we have,

$$a^2 u_{xx} = u_{tt}, \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0.$$

- Let  $\phi$  be any differentiable function. Show that  $u(x, t) = \phi(x \pm at)$  satisfy the wave equation; that is show  $a^2 \partial_{xx} \phi(x \pm at) = \partial_{tt} \phi(x \pm at)$ .
- Let  $u(x, t) = \frac{1}{2}(f(x - at) + f(x + at))$ . Show that this satisfies the wave equation and the two initial conditions.
- Let  $f(x)$  be given by the graph below. Let  $a = 1$ . Graph  $u(x, 2)$  where  $u$  is the solution given in part b. Then graph  $u(x, 3)$  and  $u(x, 4)$ . It is a traveling wave!

