9.3 **Theorem** Suppose $s_n \to s$ and $t_n \to t$ with $s, t \in \mathbb{R}$. Then $s_n + t_n \to s + t$.

Pf Let $\varepsilon > 0$.

$\exists N_1 \in \mathbb{N}$ s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$.

$\exists N_2 \in \mathbb{N}$ s.t. $n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$.

Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$|(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| < |s_n - s| + |t_n + t|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $s_n + t_n \to s + t$. $\boxtimes$

9.4 Theorem Suppose $s_n \to s$ and $t_n \to t$, with $s, t \in \mathbb{R}$. Then $s_n t_n \to st$.

This proof is quite a bit harder. We will use the fact that Thm 9.1 says convergent sequences are bounded.

Pf Let $\varepsilon > 0$.

By Thm 9.1 $\exists\, M > 0$ s.t. $|s_n| \leq M \;\; \forall n \in \mathbb{N}$.

$\exists N_1 \in \mathbb{N}$ s.t. $n > N_1 \Rightarrow |t_n - t| < \dfrac{\varepsilon}{2M}$.

$\exists N_2 \in \mathbb{N}$ s.t. $n > N_2 \Rightarrow |s_n - s| < \dfrac{\varepsilon}{2(|t|+1)}$

Let $N = \max\{N_1, N_2\}$. Then for $n > N$ we have

$$
\begin{aligned}
|s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\
&\leq |s_n t_n - s_n t| + |s_n t - st| \\
&= |s_n|\,|t_n - t| + |t|\,|s_n - s| \\
&< M \frac{\varepsilon}{2M} + |t| \frac{\varepsilon}{2|t|+1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

since $\dfrac{|t|}{2|t|+1} < \dfrac{1}{2}$.

9.7 Theorem

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

As the text notes n^p has not been defined for $p \notin \mathbb{Q}$. So, I'll only work with $p \in \mathbb{Q}$. We will need to use that $0 < a < b \Rightarrow a^p < b^p$ for $p \in \mathbb{Q} \cap (0,\infty)$. We have only shown this for $p \in \mathbb{N}$ and $p = \frac{1}{2}$. To prove this in general, let $p = \frac{1}{k}$, $k \in \mathbb{N}$. Suppose, $b^{\frac{1}{k}} \leq a^{\frac{1}{k}}$. Then $(b^{\frac{1}{k}})^k \leq (a^{\frac{1}{k}})^k \Rightarrow b \leq a$, which is false. Let $p = \frac{m}{k}$, Then

$$0 < a < b \Rightarrow a^m < b^m \Rightarrow a^{\frac{m}{k}} < b^{\frac{m}{k}}.$$

Now, the proof in the textbook (pg 48) is valid for $p > 0, p \in \mathbb{Q}$. You can read it.

9.7 Theorem (c) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

This will be used in several places later in this course.

The proof uses the Bionomial Theorem. This is an exercise on page 6 in Ross's textbook. You may have seen the proof in MATH 349. It says

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Pf of 9.7 (c). Since $n \geq 1$, $n^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1$. Let $s_n = n^{\frac{1}{n}} - 1$.
Then $s_n \geq 1$. Notice $n = (1+s_n)^n$.
we have

$$(1+s_n)^n = \binom{n}{0}1^n + \binom{n}{1}s_n + \binom{n}{2}s_n^2 + \cdots + \binom{n}{n}s_n^n = 1 + ns_n + \frac{n(n-1)}{2}s_n^2 + \cdots + s_n^n.$$

Assume $n \geq 2$. Then,

$$(1+s_n)^n \geq 1 + ns_n + \frac{n(n-1)}{2}s_n^2 \geq \frac{n(n-1)}{2}s_n^2.$$

Thus $n > \frac{n(n-1)}{2}s_n^2$.

$$\Rightarrow \frac{2}{n-1} > s_n^2. \Rightarrow s_n < \sqrt{\frac{2}{n-1}}.$$

$\lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$. Thus, $\lim_{n\to\infty} s_n = 0$. Thus $\lim_{n\to\infty} n^{\frac{1}{n}} - 1 = 0$

So $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. $\square$