<u>Abel's Theorem</u> Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R, where $0 < R < \infty$. We already know $\sum_{n=0}^{\infty} a_n x^n$ converges to a continuous function, $f(x)$, on $(-R, R)$.

If the series converges at $x = R$ and we define $f(R)$ to be this limit, then f is cont. at $x = R$.

If the series converges at $x = -R$ and we define $f(-R)$ to be this limit, then f is cont at $x = -R$.

<u>Pf</u> We will prove the result for a special case and later show how this can be generalized. Assume for now that the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is one, 1, and that at $x = 1$ the series converges to zero, 0.

Let $f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k$ for $x \in (-1, 1]$.

We know f is cont. on $(-1, 1)$. We will show the convergence is uniform on $[0, 1]$, hence f is cont. on $(-1, 1]$.

Let $f_n(x) = \sum_{k=0}^{n} a_k x^k$ and $s_n = \sum_{k=0}^{n} a_k = f_n(1)$.

Let $\varepsilon > 0$. Since $s_n \to f(1) = 0$, $\exists N$ s.t. $n \geq N \Rightarrow |s_n| < \frac{\varepsilon}{3}$.

Let $n > m \geq N$. Then

$$f_n(x) - f_m(x) = \sum_{k=m+1}^{n} a_k x^k = \sum_{k=m+1}^{n} (s_k - s_{k-1}) x^k$$

$$= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m+1}^{n} s_{k-1} x^{k-1}$$

$$= \sum_{k=m+1}^{n} s_k x^k - x \sum_{k=m}^{n-1} s_k x^k$$

$$= s_n x^n + \sum_{k=m+1}^{n-1} s_k x^k - x s_m x^m - x \sum_{k=m+1}^{n-1} s_k x^k$$

$$= s_n x^n - s_m x^{m+1} + (1-x) \sum_{k=m+1}^{n-1} s_k x^k.$$

Now,

$$|f_n(x) - f_m(x)| \leq |s_n| x^n + |s_m| x^{m+1} + \left| (1-x) \sum_{k=m+1}^{n-1} s_k x^k \right|$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} (1-x) \sum_{k=m+1}^{n-1} x^k$$

If $x = 0$ or 1, this is $= \frac{2\varepsilon}{3} < \varepsilon$.
Otherwise, use the Geometric Series formula on pg 96 to show that

$$(1-x) \sum_{k=m+1}^{n-1} x^k = x^{m+1}(1 - x^{n-m-1}).$$

This is less that 1 for x in $[0,1]$. Thus,

$$|f_n(x) - f_m(x)| < \varepsilon \quad \text{for } n > m \geq N.$$

Thus, (f_n) is uniformly Cauchy on $[0,1]$. Hence the convergence is uniform and the limit function, f, is continuous on $[0,1]$.

Now we drop the assumptions that $R=1$ and that $f(R)=0$. We have $f(x)=\sum_{n=0}^{\infty} a_n x^n$ on $(-R,R]$.

Let $g(x) = f(Rx) - f(R)$. Then

$$g(x) = \sum_{n=0}^{\infty} a_n R^n x^n - f(R)$$

We can replace a_0 with $(a_0 - f(R))$ to get a standard power series. Now the radius of convergence is 1 and $g(1)=0$. Thus, g is cont. on $\cancel{(-R,R]}$. $[0,1]$ and in fact on $(-1,1]$. Thus, f is cont. on $(-R,R]$.

The last bit (Case 3 in the textbook's version) is the case when the series converges at $x=-R$. This is handled by defining a new function $h(x)=f(-x)$. See textbook.

$\square$