Section 12
More on lim sup and lim inf

Thm 12.1 Let $s_n \to s$ where $s > 0$, and let (t_n) be any seq. Then
$$\limsup s_n t_n = s \limsup t_n.$$

pf First we will show that
$$\limsup s_n t_n \geq s \cdot \limsup t_n.$$

Let $\beta = \limsup t_n$

There are three cases: β is finite, $\beta = \infty$, $\beta = -\infty$. I'll do the first. You can read the other two in the textbook.

Case 1 Assume β is finite. We know β is a cluster pt for (t_n) [in fact the largest cl. pt.], thus by Thm 11.7 $\exists$ (t_{n_k}) s.t. $t_{n_k} \to \beta$.

Since (s_n) converges to s, so does any subseq. (Thm 11.3). Hence, $s_{n_k} \to s$.

Thus, $s_{n_k} t_{n_k} \to s\beta$.
By Thm 11.8 $s\beta \leq \limsup s_n t_n$. But this is the same as
$$\limsup s_n t_n \geq s \limsup t_n.$$

Again, see textbook for case 2 ($\beta = \infty$) and case 3 ($\beta = -\infty$).

Now, we have $\limsup s_n t_n \geq s \cdot \limsup t_n$, but we want to show they are equal. So, we will show that $\limsup s_n t_n \leq s \limsup t_n$.

Suppose for now that none of the s_n's are zero. Recall $s > 0$. Thus

$$\limsup t_n = \limsup \left(\frac{1}{s_n}\right)(s_n t_n) \geq \frac{1}{s} \limsup s_n t_n$$

Thus

$$\limsup s_n t_n \leq s \limsup t_n,$$

as desired.

If a finite number of s_n's are zero we can skip past them without affecting the lim sup's. Since, $s_n \to s > 0$, $\exists N$ s.t. $n > N \Rightarrow s_n > 0$. Hence, we are done! $\square$

Thm 12.2 Let (s_n) be a seq of nonzero real numbers. Then

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

Pf The middle $\le$ is immediate. I'll prove the first $\le$. The textbook does the third $\le$.

Let $A = \liminf \left|\frac{s_{n+1}}{s_n}\right|$ and $B = \liminf |s_n|^{1/n}$.

If $A = 0$, we are done. So, we assume $A > 0$.

If we can show that $A' \le B$ $\forall\, 0 < A' < A$, then $A \le B$.
Let $A' < A$ be arbitrary.

We know $\liminf \left|\frac{s_{n+1}}{s_n}\right| = \lim\limits_{N\to\infty} \inf\left\{\left|\frac{s_{n+1}}{s_n}\right|, n > N\right\} > A'$.

Thus, $\exists\, N$ s.t. $\inf\left\{\left|\frac{s_{n+1}}{s_n}\right|, n \ge N\right\} > A'$.

Thus, for $n \ge N$ $\left|\frac{s_{n+1}}{s_n}\right| > A'$.

Now here is the trick. For $n > N$

$$|s_n| = \underbrace{\left|\frac{s_n}{s_{n-1}}\right|\left|\frac{s_{n-1}}{s_{n-2}}\right|\cdots\left|\frac{s_{N+1}}{s_N}\right|}_{N-n \text{ terms}} |s_N| > (A')^{n-N}|s_N|.$$

Let $c = (A')^{-N} |s_N|$. Then $c > 0$ and recall that $c^{1/n} \to 1$ (Thm 9.7 d).

Now, for $n > N$, $|s_n| > c\,(A')^n$.

Thus, $|s_n|^{1/n} > c^{\frac{1}{n}} A'$.

Thus, $\liminf |s_n|^{1/n} \geq \liminf c^{\frac{1}{n}} A' = A'$. *

That is $B \geq A'$, so $B \geq A$ as claimed.

Here we are using that the first N terms do not affect the lim inf.

Also see Corollary 12.3 in the text book.