Continuous Functions
Section 17

Definition 1. Most calculus textbooks say a function f is continuous at a point c in its domain if

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x).$$

Your textbook does not define limits of functions of $\mathbb{R}$ until Section 20. Instead it gives this definition which we will later see is equivalent.

Definition 2. Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$. Then f is continuous at $c \in D$ if $\forall$ sequences in D with $x_n \to c$ we have

$$f(x_n) \to f(c).$$

We say f is continuous on D if it is continuous at each $c \in D$.

However, the "standard" definition of continuity is the following.

Definition 3. Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$. Then f is continuous at $c \in D$ if $\forall\, \epsilon >$, $\exists\, \delta > 0$ such that $\forall\, x \in D$ we have

$$|x - c| < \delta \quad \Longrightarrow \quad |f(x) - f(c)| < \epsilon.$$

We say f is continuous on D if it is continuous at each $c \in D$.

Theorem (17.2). Definitions 2 and 3 are equivalent.

Proof. ($3 \implies 2$) Assume $f : D \to \mathbb{R}$ is continuous at $c \in D$ according to Definition 3. Let $x_n \to c$ where each $x_n \in D$.

Let $\epsilon > 0$. $\exists\, \delta > 0$ s.t. $|x - c| < \delta \implies |f(x) - f(c)| < \epsilon$.

$\exists\, N$ s.t. $n > N \implies |x_n - c| < \delta$.

Thus, $n > N \implies |f(x_n) - f(c)| < \epsilon$.

Thus, $f(x_n) \to f(c)$.

($2 \implies 3$) We will assume 3 is false and show this implies 2 is false. Let $c \in D$ and suppose f is not continuous at c according to Definition 3. We state the negation of Definition 3.

$\exists\, \epsilon > 0$ s.t. $\forall \delta > 0\ \exists\, x \in D$ with $|x - c| < \delta$, but $|f(x) - f(c)| \geq \epsilon$.

Now we construct sequence (x_n) that will converge to c, but such that $(f(x_n))$ will not converge to $f(c)$.

Let $\delta = 1$. Chose $x_1 \in D$ with $|x_1 - c| < 1$, but $|f(x_1) - f(c)| \geq \epsilon$.

Let $\delta = 1/2$. Chose $x_2 \in D$ with $|x_2 - c| < 1/2$ but $|f(x_2) - f(c)| \geq \epsilon$.

Let $\delta = 1/3$. Chose $x_3 \in D$ with $|x_3 - c| < 1/3$ but $|f(x_3) - f(c)| \geq \epsilon$.

In general for each $n \in \mathbb{N}$ chose $x_n \in D$ with $|x_n - c| < 1/n$, but $|f(x_n) - f(c)| \geq \epsilon$.

Then, as you can check, $x_n \to c$ but $f(x_n) \not\to f(c)$. $\square$

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x^3+x}{x^2+1}$. Show that f is continuous on $\mathbb{R}$.

Proof. Let $c \in R$ and suppose (x_n) is a sequence in $\mathbb{R}$ such that $x_n \to c$. Using the limit theorems from Section 9 we have

$$\lim_{n\to\infty} f(x_n) = \frac{\lim_{n\to\infty} x_n^3 + \lim_{n\to\infty} x_n}{\lim_{n\to\infty} x_n^2 + \lim_{n\to\infty} 1} = \frac{c^3 + c}{c^2 + 1} = f(c).$$

□

Example. Let $f(x) = \sqrt{x}$. Then f is continuous on $[0, \infty)$. This follows from Example 5 in Section 8.

Example. Let $f(x) = \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}$
Then f is continuous everywhere except $x = 0$.

Next up is a string of theorems about combining continuous functions. Let $f : D \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be continuous. Let $k \in \mathbb{R}$.

- $|f|$ is continuous on D.
- kf is continuous on D.
- $f + g$ is continuous on $D \cap B$
- fg is continuous on $D \cap B$.
- $1/f$ is continuous on D except at points where $f(x) = 0$.
- If $f(D) \subset B$, then $g \circ f$ is continuous on D.

The proofs of these are all very easy. You should do them.

Example. Let $g(x) = \frac{2+\frac{1}{x}}{x^2-9}$ and $f(x) = \sqrt{g(x)}$. What is the domain of g. Is g continuous on its domain? What is the domain of f. Is f continuous on its domain? (By domain we mean the largest subset of $\mathbb{R}$ where the operations determine real values.)

Answer. The domain of g is $(-\infty,-3)\cup(-3,0)\cup(0,3)\cup(3,\infty)$. It is continuous on its domain. The domain of f is $(-\infty,-3)\cup[-1/2,0)\cup(3,\infty)$. It is continuous on its domain. See graphs below.