Extreme Theorems
Section 18

Definition. A subset I of $\mathbb{R}$ with more than one point is called an **interval** if whenever $a, c \in I$ and $a < b < c$ then $b \in I$.

There are nine types of intervals:

(a, b)	(a, ∞)	$\mathbb{R} = (-\infty, \infty)$
$[a, b)$	$[a, \infty)$	
$(a, b]$	$(-\infty, b)$	
$[a, b]$	$(-\infty, b]$	

It is easy though laborious to check these are the only possible types of intervals. An interval I either has $\sup I$ infinity or finite and if finite I either contains $\sup I$ or it does not. Likewise for $\inf I$. If you check all the possibilities you get the nine cases.

Intervals of the forms (a, b), $(-\infty, b)$, (a, ∞) and $\mathbb{R}$ are said to be **open intervals**. A subset of $\mathbb{R}$ is **open** if it is a union of open intervals. A subset of $\mathbb{R}$ is **closed** if it is the complement of an open subset. (The empty set, $\emptyset$, is defined to be open, which leads to the oddity that $\mathbb{R}$ and $\emptyset$ are both open and closed!)

Example. The set $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is closed since its complement

$$(-\infty, 0) \cup \left(\bigcup_{n=1}^{\infty} \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right) \right) \cup (1, \infty)$$

is open. (Draw pictures until you see this.)

Question. Is the Cantor Middle Thirds set open, closed or neither

Intervals of the form $[a, b]$ are said to be **compact** intervals. More general compact sets will be defined later.

Definition. A function $f : D \to \mathbb{R}$ is **bounded above** if $\exists\ B \in \mathbb{R}$ s.t. $f(x) \leq B$, $\forall x \in D$. A function $f : D \to \mathbb{R}$ is **bounded below** if $\exists\ B \in \mathbb{R}$ s.t. $f(x) \geq B\ \forall x \in D$. A function $f : D \to \mathbb{R}$ is **bounded** is if it is bounded above and below.

Extreme Value Theorem (18.1). Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded. Furthermore, $\exists\ c \in [a, b]$ s.t. $f(c) \geq f(x)\ \forall x \in [a, b]$ and $\exists\ d \in [a, b]$ s.t. $f(d) \leq f(x)$ $\forall x \in [a, b]$.

Examples. Before proving this theorem we give some examples showing this theorem fails if the domain is not compact or f is not continuous.

1. $\tan x$ is continuous on $(-\pi/2, \pi/2)$ but is not bounded above or below.
2. $\sec x$ is continuous on $(-\pi/2, \pi/2)$ but is not bounded above. It is bounded below and has a minimum value at $x = 0$.
3. $\arctan x$ is continuous on $\mathbb{R}$, is bounded above and below, but has no minimum or maximum values.
4. x^2 is continuous on $\mathbb{R}$, is unbounded above, but is bounded below and has a minimum value at $x = 0$.

Proof. Let $I = [a, b]$.

First we show that f is bounded. Suppose f is not bounded above. Then for each $n \in \mathbb{N}$, $\exists\, x_n \in I$ s.t. $f(x_n) > n$. Thus $f(x_n) \to \infty$. Now, consider the sequence (x_n). By the Bolzano-Weierstrass Theorem it has a convergent subsequence, $x_{n_k} \to c \in I$. By the continuity of f, $f(x_{n_k}) \to f(c)$. But this impossible. Thus, f is bounded from above.

A similar argument shows f must be bounded from below.

Now that f is bounded let $M = \sup f(I)$. ($f(I) = \{f(x) \mid x \in I\}$) For each $n \in \mathbb{N}$ $\exists$ $x_n \in I$ s.t. $M - \frac{1}{n} < f(x_n) \leq M$. By the Squeeze Theorem $f(x_n) \to M$. By the B-W Theorem (x_n) has a convergent subsequence $x_{n_k} \to c \in I$. Also $f(x_{n_k}) \to M$. Thus, by continuity $f(c) = M$.

Let $N = \inf f(I)$. A similar argument shows $\exists\, d \in I$ s.t. $f(d) = N$. □

Intermediate Value Theorem. Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be continuous. Let $a, b \in I$, $a < b$ and $f(a) \neq f(b)$. If y is in between $f(a)$ and $f(b)$, then $\exists$ $x \in (a, b)$ s.t. $f(x) = y$.

(Draw pictures in class.)

Proof. Suppose, $f(a) < y < f(b)$.

Let $S = \{x \in [a, b] \mid f(x) < y\}$. Since $a \in S$, we know $S \neq \emptyset$. Since b is an upper bound of S we know S has a least upper bound. Let $x_0 = \text{lub } S$; clearly, $x_0 \leq b$.

We will show that $f(x_0) = y$. First we show that $f(x_0) \leq y$, then we show that $f(x_0) \geq y$. Let $n \in \mathbb{N}$. Then $x_o - \frac{1}{n}$

is not an upper bound of S. Hence, $\exists$ $x_n \in S$ s.t.

$$x_0 - \tfrac{1}{n} < x_n \leq x_0.$$

Consider the sequence (x_n). By the Squeeze Theorem $x_n \to x_0$.

By continuity $f(x_n) \to f(x_0)$. Since each $f(x_n) < y$ we know $f(x_0) \leq y$. [To prove this suppose $f(x_0) > y$. Let $\epsilon = (f(x_0) - y)/2$. Then for no value of n is $f(x_n) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$. This contradicts convergence.]

Now we show $f(x_0) \geq y$. Let $n \in \mathbb{N}$. Then $x_0 + \frac{1}{n}$ is an upper bound of S. The sequence $x_0 + \frac{1}{n}$ converges to x_0, but its terms may not be in the domain of f. To get around this let $t_n = \min\{b, x_0 + \frac{1}{n}\}$. Now $t_n \to x_0$ and $f(t_n)$ is defined.

By continuity $f(t_n) \to f(x_0)$. Since t_n is never in S, $f(t_n) \geq y$. Thus $f(x_0) \geq y$.

By ordered field axiom O2, $f(x_0) = y$ and we are done. □

A Fixed Point Theorem. Let $f : [0, 1] \to [0, 1]$ be continuous. Then $\exists\, c \in [0, 1]$ s.t. $f(c) = c$.

Proof. See textbook, Example 1, pages 135-136. This would be a reasonable test question. □

Theorem (Corollary 18.3 in textbook). Let $f : D \to \mathbb{R}$ be continuous. Let $I \subset D$ be an interval. Then $J = f(I)$ is an interval or a point.

Proof. If f is a constant on I then $f(I)$ is a point. Let $a, c \in J$ and suppose $a < b < c$. We must show $b \in J$. Let $a', c' \in I$ with $f(a') = a$ and $f(c') = c$. By the IVT $\exists\, b'$ in between a' and c' with $f(b') = b$. Since $b' \in I$ it follows that $b \in J$. Thus, J is an interval. □

Different kinds of intervals can be mapped onto each other.

Let $f(x) = x/(x^2 - 1)$. Then $f((-1, 1)) = \mathbb{R}$.

Let $g(x) = x^2$. Then $g((-1, 1)) = [0, 1)$.

Let $s(x) = \sin(x)$. Then $s((-10, 10)) = [-1, 1]$.

Let $a(x) = \arctan(x)$. Then $a(\mathbb{R}) = (-\pi/2, \pi/2)$.

Let $b(x) = e^{-x^2}$. Then $b(\mathbb{R}) = (0, 1]$.

Remark. However, it is shown in MATH 452 that the continuous image of a compact interval is a compact interval or a point. The proof is actually not hard. Let $f : [a, b] \to \mathbb{R}$ be continuous. By the EVT there exists points c and d in $[a, b]$ s.t. $f(d) \le f(x) \le f(c)$ for all $x \in [a, b]$. If it happens that $f(d) = f(c)$ then $f([a, b])$ is a point. Otherwise, for any $y \in (f(d), f(c))$ the IVT asserts that there is a point x in between c and d s.t. $f(x) = y$. Thus, $f([a, b]) = [f(d), f(c)]$.

Some review on functions from MATH 302. (I probably should put this at the beginning of Ch 3 notes. It is in Ch 4 of the main MATH 302 textbook.)

Let $f : X \to Y$ be a function. The **image** of f is $f(X) = \{f(x) \in Y \mid x \in X\}$. This is also called the **codomain** of f. Many books also call this the **range** of f, but the MATH 302 textbook called all of Y the range of f. Some books call Y the **target** set of f. Do not get too hung up on the terminology. We can also define the image of a subset of the domain.

If the image of f is all of Y then f is an **onto** function. It is also said to be **surjective** which comes from a French word for onto. Given a function $f : X \to Y$ we can create an onto function $f_{onto} : X \to f(X)$ that equals f. Normally we do not even bother giving this function a new symbol and just write $f : X \to f(X)$.

If $B \subset Y$ then the **inverse image** of B is $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$. This is also called the **preimage** or sometimes the **pullback** of B. It can happen that the inverse image of a set is empty. Note that here f^{-1} can be thought of as a function from the set of subsets of Y to the set of subsets of X.

We say f is **one-to-one** or **injective** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. If $f : X \to Y$ is one-to-one and onto then it is called a **one-to-one correspondence** or a **bijection**.

In this case f^{-1} has the property that f^{-1} of a one-point set is a one-point set. Thus, we can regard it as a function from Y to X. Then $f^{-1} : Y \to X$ is given by $f^{-1}(y)$ equals the only $x \in X$ such that $f(x) = y$. We say f is an **invertible function** and that f^{-1} is the inverse of f.

Back to MATH 352.

Definition. Let $f : I \to \mathbb{R}$, where I is an interval. Let $a, b \in I$.
If $a < b$ implies $f(a) < f(b)$, then we say f is **increasing** on I.
If $a < b$ implies $f(a) > f(b)$, then we say f is **decreasing** on I.

Fact. If f is increasing on I it is one-to-one on I. If f is decreasing on I it is one-to-one on I.

Theorem. Let $f : I \to J$, where and I and J are intervals in $\mathbb{R}$. Suppose f is one-to-one and onto.
(a) If f is increasing on I, then f^{-1} is increasing on J.
(b) If f is decreasing on I, then f^{-1} is decreasing on J.

Proof. We will prove (a). The proof of (b) is similar. Let $y_1, y_2 \in J$ with $y_1 < y_2$. Let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Clearly $x_1 \neq x_2$, so either $x_1 < x_2$ or $x_2 < x_1$. But, the latter would contradict the given that f is increasing. Thus, $y_1 < y_2$ implies $x_1 < x_2$ so f^{-1} is increasing on J. $\square$

Remark. It can be shown from this that if f is continuous and has an inverse, then the inverse is also continuous. See Theorems 18.4 and 18.5 in the textbook. This will be used in Section 29 to derive a formula for the derivative of the inverse of a function. See Theorem 29.9.

Theorem (18.6). Let $f : I \to \mathbb{R}$, where I is an interval. If f is continuous and one-to-one on I, then f is either increasing or decreasing on I.

Proof. See textbook, page 138. Treat it as optional reading. $\square$