Term by Term
Section 26

Section 26 is awkward because the author will be studying derivatives and integrals of power series even though derivatives and integrals have not yet been defined.

The main results are that term-by-term differentiation and integration do not change the radius of convergence. We will be interested in showing that the convergence is uniform, and we will study behavior at the end points of the interval of convergence.

Definition. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. For now define its **derivative** to be

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

We define its **integral** to be

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

You should notice that

$$\left(\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}\right)' = \sum_{n=0}^{\infty} a_n x^n.$$

Fact (Lemma 26.3) The derivative and integral of a power series have the same radius of convergence as the original series.

Proof. We will only do the derivative case. Let R be the radius of convergence of the original series. Then one over the radius of convergence of the derivative is

$$\limsup |na_n|^{1/n} = \limsup n^{1/n}|a_n|^{1/n} = \lim n^{1/n} \cdot \limsup |a_n|^{1/n} = 1/R.$$

□

The Theorems 26.4 and 26.5 are devoid of meaning at this stage.

Theorem 26.1 Suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R > 0$. Let P be a real number s.t. $0 < P < R$. Then the power series converges uniformly on $[-P, P]$. The limit is a continuous function.

Proof. The proof is an application of the Weierstrass M-test. First, notice that $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of convergence since since the formula for the radius of convergence depends only on $|a_n|$.

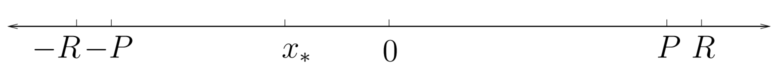
Since $|P| < R$ we know that $\sum |a_n| x^n$ converges for $x = P$, that is $\sum |a_n| P^n < \infty$.

Let $M_k = |a_k| P^k$ for $k = 0, 1, 2, 3 \ldots$. Since for $x \in [-P, P]$

$$|a_k x^k| \leq |a_k| P^k = M_k \;\; \& \;\; \sum M_k < \infty$$

the Weierstrass M-test shows the original power series converges uniformly on $[-P, P]$. Since each of the partial sums is a polynomial and polynomials are continuous, the limiting function is continuous. □

With a little more work we can show the limit is continuous on all of $(-R, R)$ even though the convergence need not be uniform. This is Corollary 26.2 in the textbook. *Proof.* Let $x_* \in (-R, R)$. If $x_* \geq 0$ let P be any point between x_* and R. If $x_* < 0$ let P be any point between $-x_*$ and R. Then $x_* \in [-P, P] \subset (-R, R)$.



The last theorem then shows the limit function is continuous at x_*.

Now we consider the end points.

Abel's Theorem (26.6). Consider a power series $\sum a_n x^n$ that has radius of convergence R s.t. $0 < R < \infty$.
If the series converges at $x = \pm R$ then the limit function is continuous there.

Example. The Taylor series of $\arctan x$ is known to be

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

with radius of convergence $R = 1$.

See: https://www.youtube.com/watch?v=Hh1VIxc9ZgM

For $x = 1$ the alternating series test shows that it converges. Abel's Theorem shows the limit is continuous. Therefore,

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan(1) = \frac{\pi}{4}.$$

This is one way to compute π. (It converges at $x = -1$ as well.)

The proof of Abel's Theorem is long and I will do it separately.