Section 28: The Derivative
Part I

Section 28 is relatively light. It is important here to not only understand the technical details of the proofs, which are not all that hard, but also to develop a sound intuition for why they make sense.

Definition. Let f be a real-valued function and assume its domain contains an open interval containing the point a. Then we say f is **differentiable** at a if the limit

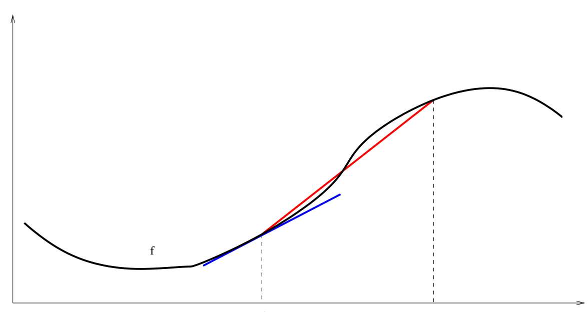
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists as a real number. The value of this limit, denoted $f'(a)$, is called the **derivative** of f at a. If f is differentiable on some set S then we can regard f' as a new function defined on S.

Motivation. The idea behind the definition is that we are interested in finding the slope of the line tangent to the curve $y = f(x)$ at the point where $x = a$. This slope is the **rate of change**. In the figure below the slope of the red line (called a secant line) is given by

$$\frac{f(x) - f(a)}{x - a}.$$

As we slide x toward a the slope will move toward the slope of the blue tangent line.



If you keep this picture in mind, you should be able to reproduce the definition of the derivative without have to memorize the formula.

Note. Many books use a different form of the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

when the limit exist as a real value. You should redraw the picture above and label it so it matches this form of the definition. This form is often easier to work with. I'll be using it.

Note. If the domain of f is of the form $[a, b]$ we can define the derivative at the end points by using one sided limits.

Note. The Leibniz notation for the derivative is $\frac{df}{dx}$. However, it is a not a ratio. The symbols df and dx do not have any meaning in isolation in calulus courses.

Next we compute some examples using the definition. Then we will derive some general rules, like the Sum Rule, the Product Rule, the Chain Rule, etc.

Example. Find the derivative of $f(x) = x^3 + 2x$ using the definition and the properties of limits.

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[(x+h)^3 + 2(x+h)] - [x^3+2x]}{h} \\ &= \lim_{h\to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h] - [x^3+2x]}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 2h}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 2 \\ &= 3x^2 + 2. \end{aligned}$$

□

Example. Find the derivative of $f(x) = \frac{1}{2x+3}$ using the definition and the properties of limits. Assume $x \neq -3/2$.

Proof.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{2(x+h)+3} - \frac{1}{2x+3}}{h} \\ &= \lim_{h\to 0} \frac{[2x+3] - [2(x+h)+3]}{[2(x+h)+3][2x+3]} \cdot \frac{1}{h} \\ &= \lim_{h\to 0} \frac{-2h}{[2(x+h)+3][2x+3]} \cdot \frac{1}{h} \\ &= \lim_{h\to 0} \frac{-2}{[2(x+h)+3][2x+3]} \\ &= \frac{-2}{[2(x)+3][2x+3]} \\ &= \frac{-2}{(2x+3)^2}. \end{aligned}$$

□

Example. Find the derivative of $f(x) = |x|$ for $x \neq 0$ and prove the derivative does not exist at $x = 0$ using the definition and the properties of limits.

Proof. Suppose $x > 0$. Then $\exists\, \delta > 0$ such that $(x - \delta, x + \delta) \subset (0, \infty)$. On this set $f(x) = x$. Thus,

$$f'(x) = \lim_{h \to 0} \frac{(x + h) - x}{h} = \lim_{h \to 0} 1 = 1.$$

Similarly for $x < 0$ you can show $f'(x) = -1$.

At $x = 0$ the left ($h \to 0^-$) and right ($h \to 0^+$) limits do not match, so the limit as $h \to 0$ does not exist. □

Before we go on to the rules for derivatives we state an important fact, Theorem 28.2 in your textbook.

Theorem. If f is differentiable at a point a, then f is continuous at a.

The proof is easy and you should study it on your own.

Rules for Derivatives.

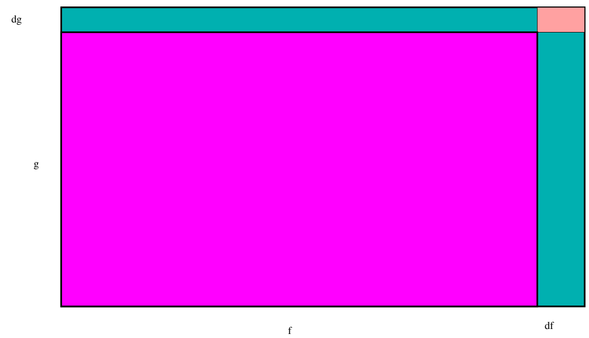
(1) The derivative of a constant function is zero, $c' = 0$.
(2) The derivative of the identity function, $f(x) = x$, is 1, $x' = 1$.
(3) $[cf(x)]' = cf'(x)$, assuming f is differentiable and c is a constant.
(4) (Sum Rule) $[f(x)+g(x)]' = f'(x)+g'(x)$, assuming f and g are differentiable.
(5) (Product Rule) $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$, assuming f and g are differentiable.
(6) (Reciprocal Rule) $[1/f(x)]' = -f'(x)/[f(x)]^2$, provided $f'(x)$ exists and $f(x) \neq 0$.
(7) (Quotient Rule) $[f(x)/g(x)]' = [f'(x)g(x) - f(x)g'(x)]/[g(x)]^2$, provided $f'(x)$ and $g'(x)$ exist and $g(x) \neq 0$.
(8) (Power Rule) $(x^n)' = nx^{n-1}$, for $n \in \mathbb{Z}$. (This works for all $n \in \mathbb{R}$ but the proof requires other means.)
(9) (Chain Rule) $[f(g(x))]' = f'(g(x))g'(x)$, assuming $f'(g(x))$ and $g'(x)$ exist.

I am going to assume you can prove 1,2, 3, and 4 on your own.

Proof of Product Rule. The form of the Product Rule should make sense to you. Notice that the "naive product rule" ($(fg)' = f'g'$) cannot be true. For one thing, the units are wrong. Suppose f and g have unit of meters and the input variable is time in seconds. Then f' and g' have units of meters/second. Thus, $f'g'$ has units of meters squared / seconds squared, while $(fg)'$ has units of meters squared / second. The units do not match! Also, this phony rule would give obviously incorrect results: $1 = (x)' = (1 \cdot x)' = 1'x' = 0 \cdot 1 = 0$.

The right way to think about the problem of finding a derivative formula for a product of two functions is to draw a picture.

Imagine that $f(x)$ and $g(x)$ represent the sides of a rectangle. Since f and g are functions, this rectangle changes as the input variable x changes. We can ask, what is the rate of change of the area of this rectangle with respect to x?



In the figure, the (lower left) purple rectangle has edge lengths $f(x)$ and $g(x)$, while the largest one has edge lengths $f(x+h)$ and $g(x+h)$. Let $\Delta f = f(x+h) - f(x)$ and $\Delta g = g(x+h) - g(x)$. Then the change in the area is the sum of the three small rectangles:

$$f(x) \cdot \Delta g + \Delta f \cdot g(x) + \Delta g \cdot \Delta f$$

To get the rate of change, just divide by h and take then limit as $h \to 0$.

$$\begin{aligned} (f(x)g(x))' &= \lim_{h\to 0} \frac{f(x) \cdot \Delta g + \Delta f \cdot g(x) + \Delta g \cdot \Delta f}{h} \\ &= \lim_{h\to 0} f(x)\frac{\Delta g}{h} + \frac{\Delta f}{h} g(x) + \frac{\Delta g}{h} \Delta f \\ &= f(x)g'(x) + f'(x)g(x) + g'(x) \cdot 0 = f(x)g'(x) + f'(x)g(x). \end{aligned}$$

The proof can be done formally without reference to a picture as is done in your textbook. But, the rectangle tells you **why** the formula makes sense. The two terms in the formula come from the two dimensions of the rectangle.