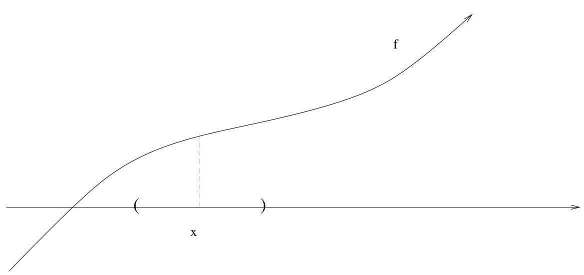
Section 28: The Derivative
Part 2

Proof of the Reciprocal Rule.

$$\begin{aligned}
\left(\frac{1}{f(x)}\right)' &= \lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h} \\
&= \lim_{h\to 0}\frac{f(x)-f(x+h)}{f(x)f(x+h)}\cdot\frac{1}{h} \\
&= \lim_{h\to 0}\frac{f(x)-f(x+h)}{h}\frac{1}{f(x)f(x+h)} \\
&= -f'(x)/[f(x)]^2.
\end{aligned}$$

But, there is a slight problem with this. We assumed $f(x)$ was not zero at this value of x. But, what if $f(x+h)$ was zero for some values of h? Here is a how we can fix this. Let $f(x) = A \neq 0$. Since f is continuous, $\forall\, \epsilon > 0\ \exists\, \delta > 0$ s.t. $|f(y) - A| < \epsilon$ for all $y \in (x-\delta, x+\delta)$. Use $\epsilon = |A|/2$. Now, $f(x+h)$ will not be zero for $h \in (-\delta, \delta)$.



Proof of the Quotient Rule.

$$(f(x)/g(x))' = f'(x)/g(x) - f(x)g'(x)/[g(x)]^2 = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Proof of the Power Rule.

For $n \in \mathbb{N}$ we use induction. You have checked $(x)' = 1 = 1x^0$. Suppose, $(x^k)' = kx^{k-1}$. Then

$$(x^{k+1})' = (x^k \cdot x)' = (x^k)'x + x^k(x)' = kx^{k-1}x + x^k = (k+1)x^k.$$

For $n < 0$ apply the Reciprocal Rule.

For $n = 0$, the rule works since $(x^0)' = 0x^{-1}$ where both sides are undefined at $x = 0$.

The Chain Rule.

Suppose Sue can run twice as fast as Bill, and Bill can run three times as fast as John. Then Sue can run six times as fast as John. That is the heart of the Chain Rule: when you compose operations the rates of change multiply.

Let $f(x) = 2x + 3$ and $g(x) = 3x - 8$. Let $h(x) = f(g(x))$. What is the rate of change of h with respect to x? Clearly, $h(x) = 2(3x-8)+3 = 6x - 13$ and has slope 6. The slopes multiply.

Even when f and g are not linear, the tangent line for $h = f \circ g$ at x will be the composition of the tangent line for g at x and the tangent line for f at $g(x)$. So, the slopes will multiply.

Here is an application. Suppose you are a deep sea diver. If you rise to the surface too fast you will get the bends (nitrogen bubbles will form in your blood vessels). Let P be pressure as a function of depth D. But, D is a function of time t. To avoid the bends you need to keep dP/dt below some thresh hold. By the Chain Rule

$$\frac{dP}{dt} = \frac{dP}{dD}\frac{dD}{dt}$$

or $[P(D(t))]' = P'(D(t))D'(t)$. Thus, you need to keep $D'(t)$ small.

I'll give a somewhat naive proof of the Chain Rule that has a gap in it. Then we will fill in the gap. Let $z = f(y)$, $y = g(x)$ and consider $f(g(x))$. Then

$$\begin{aligned}
[f(g(x))]' &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}
\end{aligned}$$

The second limit is clearly $g'(x)$. We'd like to say the first limit is $f'(g(x))$. But, this may fail. It could be the $g(x+h) - g(x) = 0$ for values of h arbitrarily close to 0. This would happen, for example, if g was a constant function. Unfortunately, the fix is not that easy.

Formal Proof of the Chain Rule.

The proof below is from Stewart's Calculus textbook in an appendix on proofs. (Which appendix depends on the edition you have.) Your book has a different proof. You can study it as well.

Chain Rule. Assume $g(x)$ is differentiable at $x = a$ and that $f(u)$ is differentiable at $u = g(a)$. Let $h(x) = f(g(x))$. Then h is differentiable at $x = a$ and $h'(a) = f'(g(a))g'(a)$.

Proof. Let $b = g(a)$. Let $\delta > 0$ be small enough that $(a - \delta, a + \delta)$ is in the domain of $h(x)$. Assume $|\Delta x| < \delta$.

Let $\Delta u = g(a + \Delta x) - g(a)$.
Let $\Delta y = f(b + \Delta u) - f(b)$.

We define two functions, $\sigma_1(\Delta x)$ and $\sigma_2(\Delta u)$.

$$\sigma_1(\Delta x) = \begin{cases} \frac{\Delta u}{\Delta x} - g'(a) & \text{for} \quad \Delta x \neq 0 \\ 0 & \text{for} \quad \Delta x = 0 \end{cases}$$

$$\sigma_2(\Delta u) = \begin{cases} \frac{\Delta y}{\Delta u} - f'(b) & \text{for} \quad \Delta u \neq 0 \\ 0 & \text{for} \quad \Delta u = 0 \end{cases}$$

Now we can express Δu and Δy as follows.

$$\Delta u = (\sigma_1(\Delta x) + g'(a))\Delta x.$$

$$\Delta y = (\sigma_2(\Delta u) + f'(b))\Delta u = (\sigma_2(\Delta u) + f'(b))(\sigma_1(\Delta x) + g'(a))\Delta x.$$

Now we put it all together.

$$h'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (\sigma_2(\Delta u) + f'(b))(\sigma_1(\Delta x) + g'(a)) = f'(b)g'(a).$$

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