Section 29: The Mean Value Thm

The Mean Value Theorem (MVT) is one you encountered in calculus but may not remember. It sort of hides in the background. But, it is a vital component of the theory of calculus.

Here is a rough statement of it: If Grandma lives 100 miles away and you drive there in one hour, she knows you were speeding! The point is if you drove 100 miles in one hour, your average speed was 100 mph. Hence at some point you were driving 100 mph which is over the speed limit. We turn this deep insight of Grandma's into math as follows.

The Mean Value Theorem. Let f be a continuous function on a compact interval $[a, b]$ that is differentiable on (a, b). Then $\exists$ at least one value $x \in (a, b)$ such that the rate of change of f at x is equal to the average rate of change of f over $[a, b]$:

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

We will build up to the proof by proving two preliminary theorems. At several points the book invokes Corollary 20.7, so I'll review that first.

Useful Fact (Cor. 20.7). Let $\lim_{x \to a} f(x) = L$. This presupposes that f is defined for values of x in an open interval containing a, but not necessarily at $x = a$. Then $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

(In fact this is and if and only if, but we only need one direction. It is just a restatement of the definition of a limit.)

Theorem. (29.1) This is often called Fermat's Theorem. If $f(x)$ has a local maximum or minimum at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.

Proof. Assume $f(x)$ has a local maximum at $x = c$. The other case is similar. Let $\delta_1 > 0$ be s.t $|c - x| < \delta_1$ implies $f(x) \le f(c)$. Suppose $f'(c) > 0$. We shall deduce a contradiction. We know

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

Let $\epsilon = f'(c)/2$. $\exists\ \delta_2 \in (0, \delta_1)$ s.t. $0 \neq |h| < \delta_2$ implies

$$\frac{f(c+h) - f(c)}{h}$$

is within ϵ of $f'(c)$ and hence

$$\frac{f(c+h) - f(c)}{h} > 0.$$

This has to be true whether h is positive or negative. But, for $h > 0$ this implies $f(c+h) > f(c)$. Contradiction!

If we had $f'(c) < 0$ a similar contradiction arises. Check this. □

Here is an outline of all the cases and subcases of the proof of Fermat's Theorem

Case I. $f(x)$ has a local maximum at $x = c$.
- (a) Suppose $f'(c) > 0$. Get contradiction.
- (b) Suppose $f'(c) < 0$. Get contradiction.

Case II. $f(x)$ has a local minimum at $x = c$.
- (a) Suppose $f'(c) > 0$. Get contradiction.
- (b) Suppose $f'(c) < 0$. Get contradiction.

We only did I(a). It is important that you see the "global structure" of the proof. It would be a good exercise for you to write out the details of each case.

Rolle's Theorem. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then $\exists$ at least one value $c \in (a, b)$ such that $f'(c) = 0$.

Proof. We know from the Extreme Value Theorem that f obtains its minimum and maximum values: $\exists\ x_1$ and x_2 in $[a, b]$ s.t.

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a, b]$. If x_1 and x_2 are the end points of $[a, b]$ then the min and max of f are equal since $f(a) = f(b)$. In this case $f'(x) = 0$ everywhere.

So, suppose either the max or min occurs at some interior point $c \in (a, b)$. Then by Fermat's Theorem $f'(c) = 0$. □

Proof of the Mean Value Theorem. See the picture in your book on page 234. The points $(a, f(a))$ and $(b, f(b))$ determine a line. An equation for this line is

$$h(x) = \left(\frac{f(b) - f(a)}{b - a}\right) \cdot (x - a) + f(a).$$

Define a new function

$$g(x) = f(x) - h(x).$$

Then g is continuous on $[a, b]$, differentiable on (a, b) and $g(a) = g(b)$ since they are both zero. Thus, we can apply Rolle's Theorem to g; $\exists$ $c \in (a, b)$ s.t $g'(c) = 0$. Hence

$$f'(c) = g'(c) + h'(c) = 0 + \frac{f(b) - f(a)}{b - a},$$

as desired. $\square$

The figure on the left illustrates Rolle's Theorem while the figure on the right illustrates the Mean Value Theorem.