Section 29: The Mean Value Thm, continued

Corollary 29.4. Let f be differentiable in (a, b) and suppose $f'(x) = 0$ on (a, b). Then f is a constant function.

Proof. Suppose f is not constant. Then $\exists$ values x_1 and x_2 in (a, b) for which $f(x_1) \neq f(x_2)$. By the MVT $\exists$ c in between x_1 and x_2 s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts that $f'(x)$ is always zero on (a, b). Thus, f must be a constant function. □

Corollary 29.5. Let f and g be differentiable functions on (a, b). If $f' = g'$ on (a, b) then for some constant $c \in \mathbb{R}$ we have $f(x) = g(x) + c$ for all $x \in (a, b)$.

Proof. Since $(f - g)' = 0$ we know $f(x) - g(x) = c$ for some constant c by the previous Corollary. □

Corollary 29.7. Let f be differentiable on (a, b).
(a) If $f'(x) > 0$ on (a, b), then f is increasing on (a, b).
(b) If $f'(x) < 0$ on (a, b), then f is decreasing on (a, b).

Proof. (a) Let $a < x_1 < x_2 < b$. Apply the MVT to f over $[x_1, x_2]$ to get an $x \in (x_1, x_2)$ s.t.

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(x) > 0$ and $x_2 - x_1 > 0$ we see that $f(x_2) - f(x_1) > 0$. Hence,

$$x_1 < x_2 \implies f(x_1) < f(x_2),$$

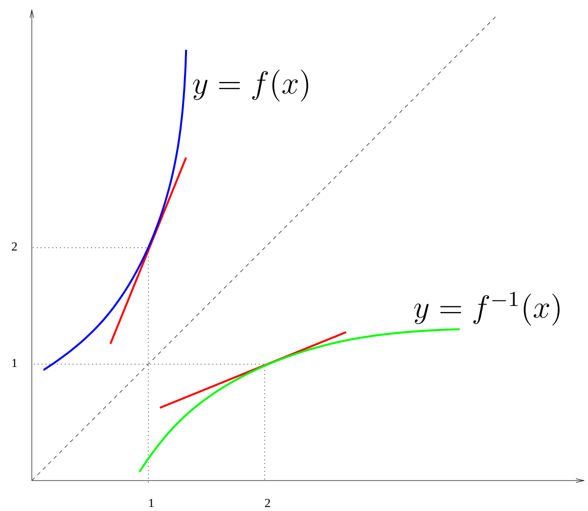
that is, f is increasing.

The proof of (b) is similar. □

Derivatives of Inverse Functions.

Example. Let $y = f(x) = 3x + 2$. Clearly, $f'(x) = 3$. Now $f^{-1}(x) = \frac{1}{3}x - \frac{2}{3}$. Now, $(f^{-1}(x))' = \frac{1}{3}$.

Example. Let $f(x)$ be the function in the graph below in blue. It is increasing and hence one-to-one so its inverse exists. The inverse is graphed in green. The slope of the tangent line to $y = f(x)$ at $x = 1$ looks to be about 3 and $f(1) = 2$. The slope of the tangent line of $y = f^{-1}(x)$ at $x = 2$ is 1/3.



Example. Let $y = x^{\frac{1}{3}}$. Find y'.

$$\begin{aligned} y^3 &= x \\ 3y^2y' &= x' = 1 \text{ by the Chain Rule} \\ y' &= \tfrac{1}{3}y^{-2} \\ y' &= \tfrac{1}{3}x^{-\frac{2}{3}} \end{aligned}$$

However, there is a problem. We assumed y' existed before we found it. The next theorem will justify this step. Note that in this example $y'(x)$ does not exist at $x = 0$. The next theorem will take care of this. I found a version in a different book that I liked better than the proof in your book. It is in a separate document because I just copied it as I did not feel like retyping it.