Section 30: L'Hospital's Rule

Example. Let $f(x) = 3x - 3$ and $g(x) = 2x - 2$. Find the limit

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{3x-3}{2x-2}.$$

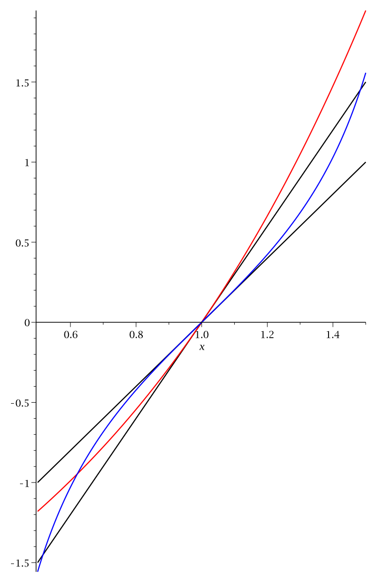
We notice that for $x \neq 1$ we have $f(x)/g(x) = 3/2$. Thus the limit is $3/2$. But this is just the ratio of the slopes. This observation is the essence of L'Hospital's Rule. However, this example is not that deep as the limit would be the same for x tending toward any value.

Example. Let $f(x) = 3(e^{x-1} - 1)$ and $g(x) = \tan(2x - 2)$. Find Find the limit

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{3(e^{x-1}-1)}{\tan(2x-2)}.$$

Check that $f(1) = g(1) = 0$. You should check that the respective tangent lines of the graphs $y = f(x)$ and $y = g(x)$ at $x = 1$ are $y = 3x - 3$ and $y = 2x - 2$.

Study the graph below. The red curve is $y = f(x)$ and the blue curve is $y = g(x)$. The two black lines are their tangent lines.



The idea behind L'Hospital's Rule is that when $f(x)$ and $g(x)$ are going to zero as $x \to 1$ (in this example) the limit of $f(x)/g(x)$ can be found by replacing the functions with their derivatives. That is,

$$\lim_{x\to 1} \frac{f(x)}{f(x)} = \lim_{x\to 1} \frac{f'(x)}{g'(x)} = \lim_{x\to 1} \frac{3e^{x-1}}{2\sec^2(2x-2)} = \frac{3e^0}{2\sec^2 0} = \frac{3}{2}.$$

Here is a simple version of L'Hospital's Rule.

Theorem. Let f and g be differentiable functions on an open interval (a, b). Suppose $c \in (a, b)$, $f(c) = g(c) = 0$ and $g'(c) \neq 0$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

Proof.

$$\frac{f'(c)}{g'(c)} = \frac{\lim\limits_{x \to c} \dfrac{f(c) - f(x)}{c - x}}{\lim\limits_{x \to c} \dfrac{g(c) - g(x)}{c - x}} = \lim_{x \to c} \frac{0 - f(x)}{0 - g(x)} = \lim_{x \to c} \frac{f(x)}{g(x)}.$$

□

There is a stronger version where we no longer assume $g'(c) \neq 0$. This allows for the possibility that the limit of $f(x)/g(x)$ is infinite or that both $f'(c) = g'(c) = 0$ and we can apply L'Hospital's Rule to the limit of $f'(x)/g'(x)$.

Theorem. Let f and g be differentiable functions on an open interval (a, b). Suppose $c \in (a, b)$, $f(c) = g(c) = 0$ and that $\lim\limits_{x \to c} f'(x)/g'(x)$ exists and $g'(x) \neq 0$ on $(c - \delta, c + \delta) - \{c\}$ for some $\delta > 0$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

The proof is unfortunately much harder. It uses another theorem called the Generalized Mean Value Theorem. Review the Rolle's Thm and the MVT before continuing if you need to.

The Generalized Mean Value Theorem. Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Then $\exists\ c \in (a, b)$ s.t.

$$f'(c)\left(g(b) - g(a)\right) = g'(c)\left(f(b) - f(a)\right).$$

Pause the video and check that if $g(x) = x$ then this gives the MVT.

Proof. Let $h(x) = f(x)\left(g(b) - g(a)\right) - g(x)\left(f(b) - f(a)\right).$

We will apply Rolle's Theorem to h. Notice that h is continuous on $[a, b]$ and differentiable on (a, b). We compute as follows.

$$h(a) = f(a)g(b) - f(a)g(a) - g(a)f(b) + g(a)f(a) = f(a)g(b) - g(a)f(b).$$

$$h(b) = f(b)g(b) - f(b)g(a) - g(b)f(b) + g(b)f(a) = -f(b)g(a) + g(b)f(a).$$

Notice $h(a) = h(b)$. Thus, by Rolle's Thm $\exists\ c \in (a, b)$ s.t. $h'(c) = 0$.

Finally,

$$0 = h'(c) = f'(c)\left(g(b) - g(a)\right) - g'(c)\left(f(b) - f(a)\right).$$

Thus,

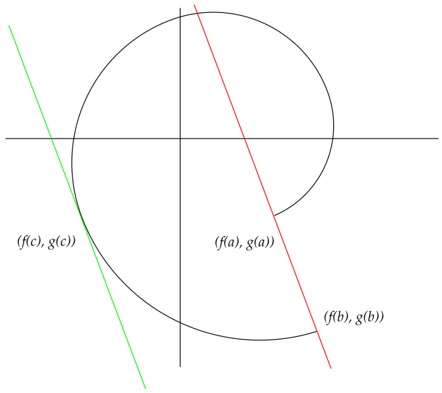
$$f'(c)\left(g(b) - g(a)\right) = g'(c)\left(f(b) - f(a)\right).$$

□

Wikipedia gives a geometrical meaning of the GMVT:
`https://en.wikipedia.org/wiki/Mean_value_theorem`
We can rewrite the result as

$$\frac{g'(c)}{f'(c)} = \frac{g(b) - g(a)}{f(b) - f(a)},$$

assuming neither denominator is zero. Now imagine that f and g determine a parametric curve $\langle f(t), g(t)\rangle$ for $t \in [a, b]$ like in the figure below from Wikipedia. The slope of the (red) line determined by the two end points is $\frac{g(b)-g(a)}{f(b)-f(a)}$. The GMVT asserts there is a value $c \in (a, b)$ where the slope of the tangent line (green) has this slope. (There are some caveats.)



Proof of the Strong Version of L'Hospital's Theorem. First we study the limit as $x \to c^+$. $\exists$ a $\delta > 0$ s.t. $g'(x) \neq 0$ for $x \in (c, c+\delta)$ by assumption. We can assume $c + \delta < b$.

We have that f and g are continuous on $[c, c+\delta]$ and differentiable on $(c, c+\delta)$. By the GMVT $\exists\, d \in (c, c+\delta)$ such that

$$f'(d)\left(g(c+\delta) - g(c)\right) = g'(d)\left(f(c+\delta) - f(c)\right).$$

Thus,

$$\frac{f'(d)}{g'(d)} = \frac{f(c+\delta) - f(c)}{g(c+\delta) - g(c)} = \frac{f(c+\delta)}{g(c+\delta)}.$$

Now, think is d as a function of δ. Notice $c < d < c + \delta$. Hence, as $\delta \to 0^+$, $d \to c^+$. Thus,

$$\lim_{\delta \to 0+} \frac{f(c+\delta)}{g(c+\delta)} = \lim_{d \to c+} \frac{f'(d)}{g'(d)}$$

This is equivalent to

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)}.$$

A similar argument shows

$$\lim_{x \to c^-} \frac{f(x)}{g(x)} = \lim_{x \to c^-} \frac{f'(x)}{g'(x)}.$$

□

What if we are interested in limits as $x \to \infty$? Then we can show L'Hospital's Rule is still valid as follows.

Let $t = 1/x$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f(1/t)}{g(1/t)}.$$

We apply L'Hospital's Rule to this limit.

$$\lim_{t\to 0^+} \frac{f(1/t)}{g(1/t)} = \lim_{t\to 0^+} \frac{[f(1/t)]'}{[g(1/t)]'} = \lim_{t\to 0+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} = \lim_{t\to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

The argument is the similar for $x \to -\infty$.

There is one more case we will consider. Suppose $f(x)$ and $g(x)$ go to infinity as $x \to c$. We will first give a proof that requires some simplifying assumptions.

Theorem. Let f and g be functions whose limits are ∞ as $x \to c$. Suppose $\lim\limits_{x\to c} f(x)/g(x) = L$ and L is neither infinite nor zero and that $\lim\limits_{x\to c} f'(x)/g'(x) = P$ is neither infinite nor zero. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Proof.

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{x\to c} \frac{1/g(x)}{1/f(x)} \\
&= \lim_{x\to c} \frac{(1/g(x))'}{(1/f(x))'} \\
&= \lim_{x\to c} \frac{-g'(x)/(g(x))^2}{-f'(x)/(f(x))^2} \\
&= \lim_{x\to c} \frac{g'(x)}{f'(x)} \left(\frac{f(x)}{g(x)}\right)^2 \\
&= \lim_{x\to c} \frac{g'(x)}{f'(x)} \left(\lim_{x\to c} \frac{f(x)}{g(x)}\right)^2
\end{aligned}$$

Hence

$$L = \frac{1}{P} L^2.$$

Thus, $P = L$. ∎

There is a trick to get the result when $L = 0$. I saw this is some lecture notes posted by Prof. Lorenzo Sadun of U.T. Austin. Consider that

$$\lim_{x \to c} \frac{f(x) + g(x)}{g(x)} = 1.$$

Then

$$\lim_{x \to c} \frac{f(x) + g(x)}{g(x)} = \lim_{x \to c} \frac{f'(x) + g'(x)}{g'(x)} = 1 + P.$$

thus, $P = L = 0$.

If $L = \pm\infty$ work with $g(x)/f(x)$ instead.

(The version in your text is still stronger then this in that it does not assume as much. But this is good enough for us.)