Examples of L'Hospital's Rule

minus

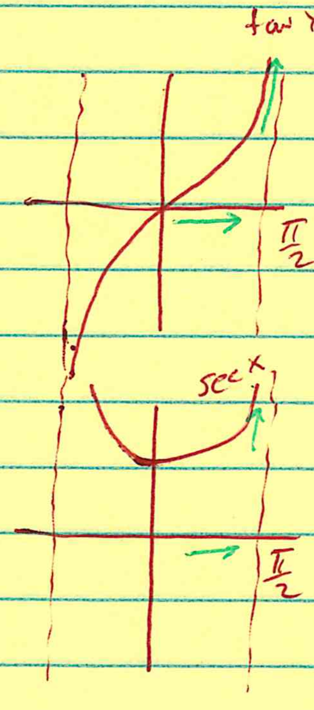
1. Find $\lim_{x \to 0} \frac{\cos x + 2x \cancel{+} 1}{3x}$.

Solution: $\cos(0) + 2 \cdot 0 - 1 = 1 - 1 = 0$ and $3 \cdot 0 = 0$.

$$\lim_{x \to 0} \frac{\cos x + 2x - 1}{3x} = \lim_{x \to 0} \frac{(\cos x + 2x - 1)'}{(3x)'} =$$

$$= \lim_{x \to 0} \frac{-\sin x + 2}{3} = \frac{-0 + 2}{3} = \frac{2}{3}$$

2. Find $\lim_{x \to \frac{\pi}{2}^-} \frac{4 \tan x}{1 + \sec x}$.



Solution: $\lim_{x \to \frac{\pi}{2}^-} 4 \tan x = \infty$

$\lim_{x \to \frac{\pi}{2}^-} 1 + \sec x = \infty$.

$$\lim_{x \to \frac{\pi}{2}^-} \frac{4 \tan x}{1 + \sec x} = \lim_{x \to \frac{\pi}{2}^-} \frac{4 \sec^2 x}{\sec x \tan x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{4 \sec x}{\tan x} = \lim_{x \to \frac{\pi}{2}} \frac{4}{\sin x} = \frac{4}{\sin(\frac{\pi}{2})} = 4.$$

3. Find $\lim\limits_{x\to\infty} \frac{\ln x}{\sqrt{x}}$.

Solution: $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to\infty} \sqrt{x} = \infty$.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to\infty} \frac{2}{x^{\frac{1}{2}}} = 0.$$

4. Prove that for all $n \in \mathbb{N}$, $\lim\limits_{x\to\infty} \frac{e^x}{x^n} = \infty$.

(This means e^x grows faster than any polynomial.)

Proof. Let $n=1$. $\lim\limits_{x\to\infty} \frac{e^x}{x} = \lim\limits_{x\to\infty} \frac{e^x}{1} = \infty$.

Suppose for a fixed k we had $\lim\limits_{x\to\infty} \frac{e^x}{x^k} = \infty$.

Let $n = k+1$.

$$\lim_{x\to\infty} \frac{e^x}{x^{k+1}} = \lim_{x\to\infty} \frac{e^x}{(k+1)x^k} = \frac{1}{k+1} \lim_{x\to\infty} \frac{e^x}{x^k} = \infty.$$

5 Find $\lim\limits_{x\to 0} \frac{x - \sin x}{x^3}$.

Solution: This is of the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} \text{ (still } \tfrac{0}{0}\text{)}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \text{ (still } \tfrac{0}{0}\text{)} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

6. Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)$

Solution: This is of the form $\infty-\infty$. L'Hospital's Rule does not apply. But

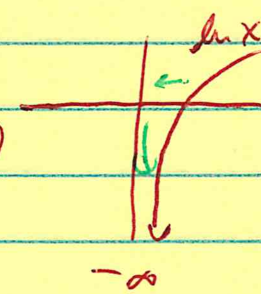
$$\frac{1}{e^x-1}-\frac{1}{x}=\frac{x-e^x+1}{xe^x-x}$$

Now as $x\to 0$ we have $\frac{0}{0}$. Thus,

$$\lim_{x\to 0}\frac{(x-e^x+1)'}{(xe^x-x)'}=\lim_{x\to 0}\frac{1-e^x}{e^x+xe^x-1} \quad (\text{still } \tfrac{0}{0})$$

$$=\lim_{x\to 0}\frac{-e^x}{e^x+xe^x+e^x}=\frac{-e^0}{e^0+0+e^0}=-\frac{1}{2}.$$

7. Find $\displaystyle\lim_{x\to 0^+}x^2\ln x$.



Solution: This is of the form $0(-\infty)$

Rewrite as $\displaystyle\lim_{x\to 0^+}\frac{\ln x}{\left(\frac{1}{x^2}\right)}$. Now it is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x^2}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-2}{x^3}}=\lim_{x\to 0^+}\frac{-x^2}{2}=0.$$

8. Find the $\lim_{x \to 0} (1+3x)^{\frac{1}{2x}}$.

Solution: This is of the form 1^{∞}.
We use a trick. We assume $\ln x$ is continuous and has the properties you learned in precalc.
Let

$$y = (1+3x)^{\frac{1}{2x}}.$$

$$\ln y = \ln (1+3x)^{\frac{1}{2x}} = \frac{\ln(1+3x)}{2x}$$

As $x \to 0$ we see $\ln y$ is of the form $\frac{0}{0}$.

Thus $$\lim_{x \to 0} \frac{\ln(1+3x)}{2x} = \lim_{x \to 0} \frac{\frac{1}{1+3x} \cdot 3}{2} = \frac{3}{2} \lim_{x \to 0} \frac{1}{1+3x}$$

$$= \frac{3}{2} \cdot 1 = \frac{3}{2}.$$

Thus $$\lim_{x \to 0} y = \lim_{x \to 0} e^{\ln y} = e^{\lim \ln y} = e^{\frac{3}{2}}.$$

Note: These are from Calculus with Analytic Geometry by Earl Swokowski, 1975.